Dimitri Leemans
Maps, polytopes and configurations

1 Individual Project's contribution to the CRP

1.1 Aims and Objectives

The Belgian school of Incidence Geometry has a long standing tradition of excellence. The work of Jacques Tits on the theory of buildings [20], made him an Abel prize winner in 2008. The Handbook of Incidence Geometry [1], edited by Francis Buekenhout in 1995, is a major reference in the field. These incidence geometries are multipartite labeled graphs. Therefore, they cover most geometric structures, including graphs, maps, polytopes, etc.

As a student of Francis Buekenhout, I have first focused my research on incidence geometry constructed from sporadic simple groups. I have been using extensively computational algebra packages as Cayley and its successor, Magma, in order to explore geometric structures. This lead me to start developing the packages on incidence geometries and coset geometries that are available nowadays in the Magma distribution. These package are implemented in the C kernel of Magma in order to have as much efficiency as possible.

The development of algorithms to check basic and not so basic properties in incidence geometries and coset geometries has led to the building of several atlases, namely atlases of primitive or residually weakly primitive geometries [2, 3, 14], an atlas of thin regular geometries [12], an atlas of polytopes for small almost simple groups [18].

My interest in polytopes started in 2002 during a collaboration with Michael I. Hartley. Our first aim was to study a locally projective polytope of Schäfli type {5,3,5} whose automorphism group was the first group of Janko J_1. We both had found this polytope independently. We found out that the universal locally projective polytope of type {5,3,5} has automorphism group $J_1 \times PSL(2, 19)$ [9]. This was a big step missing in the classification of all rank four locally projective polytopes [6]. After that, we investigated further the thin rank four geometries I obtained in 1999 while I was building my atlas of thin regular geometries [12]. We were able to reconstruct all six rank four thin geometries of J_1 starting from the {5,3,5} polytope [8]. Moreover, we obtained a new Petrie-like construction [10]. Recently, with Hartley and Hubard, we gave a geometric construction of the polytopes of Schläfli type $\{5, 3, 5\}$ and $\{5, 6, 5\}$ associated to J_1 [7].

With Hartley, we decided in 2005 to build atlases of polytopes. Hartley has classified all polytopes with less than 2000 faces [5] and I built an atlas of polytopes for small almost simple groups with one of my students [18]. This atlas is available on my website. The experimental results quickly gave rise to several theoretical results on the Suzuki simple groups [13, 11] and the almost simple groups of type $PSL(2, q)$ [16, 17]. The results obtained with Egon Schulte prove a conjecture made after the experimental results were obtained, namely that there are only two rank four polytopes whose automorphism group is a group $PSL(2, q)$: Grünbaum's 11-cells and Coxeter's 57-cells.

Here are the main themes we want to give contributions to in this project.

Computational Incidence Geometry We plan to continue the development of the incidence geometry and coset geometry packages available in Magma. Our main contributions will include new functionalities to deal with string C-groups, abstract regular and chiral polytopes, maps, hypermaps and locally s-arc-transitive graphs. Some of the functions we intend to develop include

- testing the intersection property in abstract regular polytopes and abstract chiral polytopes;
- computing genuses of maps and polyhedra;

- testing locally (G, s)-arc-transitivity of a graph.

For instance, we will make available the programs that check is a graph is locally (G, s)-arc-transitive and that we used extensively in [15].

We also plan to port the functionalities currently available in Magma to other systems like Gap, Sage, etc.

Abstract regular and chiral polytopes Our main aim is to try to make a major contribution to a question asked by Egon Schulte (AP): are there more regular or chiral polytopes ?

Roughly speaking, regular polytopes are those polytopes whose automorphism group has a unique orbit on the flags (that is the chains of maximal length), while flags of chiral polytopes split in two orbits under the action of their automorphism group. It is only recently that chiral polytopes of arbitrary rank have been constructed. These chiral polytopes are much harder to construct than regular polytopes in general. As we did for the regular polytopes, our aim will be to build up an atlas of chiral polytopes whose automorphism group is an almost simple group. Then using the experimental results obtained, we will try, as in the regular case, to obtain classifications of chiral polytopes for (infinite) families of almost simple groups. We will look at the Suzuki simple groups and their automorphism groups, the Ree groups and their automorphism groups, the groups of $PSL(2, q)$ type, and the alternating and symmetric groups.

A way to attack the question of Schulte is to try to count, up to isomorphism, how many regular/chiral polytopes have a given group G as automorphism group, for G in a family of groups. We will focus on families of almost simple groups.

For the Suzuki groups, the regular case is well known : as seen in [13], there is no regular polytope having a group G such that $Sz(q) < G \leq Aut(Sz(q))$ as full automorphism group. Moreover, for $G := Sz(q)$, the only possible rank is three. Recently, with Kiefer, we were able to compute how many polyhedra does a group $Sz(q)$ have up to isomorphism and duality [11]. In the chiral case, work is in progress with Isabel Hubard (UNAM) and Maxime Gheysens (MSc student).

Observe that Sah [19] and Conder et al. [4] have computed, up to isomorphism, the number of regular hypermaps on which a group of type $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$ acts as a regular automorphism group. We are currently investigating abstract regular polytopes for almost simple groups of $PSL(2, q)$ type with Egon Schulte (AP), Francis Buekenhout, Isabel Hubard, Daniel Pellicer (UNAM) and Julie De Saedeleer (PhD student).

For the Ree groups, work is currently in progress with Egon Schulte (AP) and Hendrik Van Maldeghem in the regular and chiral case.

For the alternating and symmetric groups, work is in progress with Maria-Elisa Fernandez (post-doc) and Ann Kiefer (MSc student) in the regular case.

Database of geometric objects on the world wide web We have already developed a web server in Brussels that permits to an internet user to launch remotely a Magma program that check properties on coset geometries stored in a database. We want to extend this web server by making it possible to connect to other computational algebra systems like Gap, Sage, Cocoa, Discreta, etc. Also, currently, coset geometries are stored in the database. We want to make the system available to more geometric structures like polytopes, graphs, designs, etc.

1.2 Methodologies

We will use methods of group theory, combinatorics, number theory, graph theory, computer science and algebra. **Computational incidence geometry:** study of literature, developing theory, mathematical proving. **Abstract regular and chiral polytopes:** study of literature, computer programming, testing and improving; proving useful structural properties; classifying polytopes for families of almost simple groups. **Database of geometric objects:** algorithms, programming, software design.

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