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Representations of symmetric graphs

1 Individual Project's contribution to the CRP

1.1 Aims and Objectives

Vertex-transitive graphs in general but also their relations to geometric objects have been an active topic of research for a long time now. Much of this interest is due to their suitability to model scientific phenomena when symmetry is an issue. It is the aim of this IP to deal with open problems concerning structural properties of vertex-transitive graphs in particular to those satisfying certain additional group-theoretic properties such as arc-transitive, half-arc-transitive or semisymmetric group actions, with emphasis to their link to open problems in the theory of configurations, theory of symmetric maps and relevant representations. In addition, IP also aims to use certain geometric representations as a tool for solving open problems in the theory of vertex-transitive graphs such as the well known open problem about the existence of Hamilton paths/cycles in vertex-transitive graphs.

Symmetric graphs - structural properties relevant for representations The research of arc-transitive graphs (and cubic arc-transitive graphs in particular) as well as half-arc-transitive graphs (quartic half-arc-transitive graphs) have received considerable attention over the years, the aim being to obtain structural results and possibly a classification of such graphs of different transitivity degrees, particular orders or satisfying additional properties (see, for example [6, 7, 16, 18, 19, 20, 22, 25, 24]).

The frequently used methods in this respect are based on covering graph techniques while using a particular additional condition of the automorphism groups of such graphs such as (im)primitivity or the existence of particular semiregular automorphisms. The aim of this IP is to carry out further research along these lines considering the importance of symmetries in representations of graphs and other incidence structures.

Graph covering algorithms An indispensable tool when studying symmetric graphs are graph covers. Graph covers emerged forty years ago in the context of maps on surfaces initially as a geometrical (topological) tool later becoming more of algebraic character and was used for classification of certain families of connected symmetric graphs which with this method splits into finding the 'basic' graphs and classifying their covers. For the former, finding the 'basic' graphs is in many instances obtained using the classification of finite simple groups (CFSG). As for the latter, the method (algorithm) of lifting automorphisms is used, see [11]. A combinatorial approach to this problem was given by Malnič, Nedela and Škoviera in [15]. The particular case of elementary abelian covers was investigated in [14]. In the past 15 years, these and similar methods were successfully applied in a number of papers dealing with enumeration, classification and construction of infinite families of graphs with specific symmetry properties.

Along these lines we propose to develop methods that avoids using CFSG for finding 'base' graphs, at least in (not so restrictive [17]) case when automorphism groups of graphs in question have an abelian semiregular subgroup, to consider the lifting conditions from the algorithmic point of view and to implement the combinatorial methods of lifting automorphisms as computer packages, e.g. in MAGMA. Due to geometrical aspects of covering techniques such an algorithm will be important in representation constructing algorithms that use base graph representations to obtain cover graph representations.

Symmetric graphs and configurations Vertex-transitive graphs have many applications, sometimes in quite surprising ways. One of such applications concerns configurations. Structural results on vertex-transitive graphs are useful in the context of symmetric configurations because there is one-to-one correspondence between bipartite vertex-transitive graphs of girth at least 6 and the so called Levi graphs of configurations, in particular Levi graphs of self-dual, point- and line-transitive combinatorial configurations. Several results making use of this correspondence are known, see [8, 21, 23].

One of many interesting questions concerning symmetric configurations is the question about the existence of weakly flag transitive (v_r) configuration [21]. Our aim here will be to investigate the possibility of obtaining a complete solution to this problem is the quartic case, which is in fact a special case of the problem about the classification of quartic half-arc-transitive graphs.

Symmetric graphs and maps A map M is an embedding of a finite connected graph X into a surface so that it divides the surface into simply-connected regions, called the faces of M.

This IP aims to study symmetric embeddings of graphs, in particular which symmetries of a given graph are also symmetries of the associate map. One of our goals is to classify regular Cayley maps over dihedral groups. The most significant result in this context is the classification of regular Cayley maps over cyclic groups due to Conder et al., see [4]. As a first step, we have recently classified regular Cayley maps over dihedral groups of order twice an odd number, see [13]. In this process we plan intensive collaboration in with PI Pisanski, PI Leemans and PI koviera.

Furthermore, this IP aims to generalize an innovative method, known as "Hamilton trees on surfaces approach" [10, 9], to show the existence of Hamilton paths/cycles in cubic Cayley graphs arising from groups with $(2, s, t)$–presentation and more generally in cubic vertex-transitive graphs.

Symmetric torusenes Following the discovery and synthesis of spheroidal fullerenes, a natural question arises as to whether there exist torus-shaped graphite-like carbon structures – torusenes. Unlike ordinary fullerenes that need the presence of twelve pentagonal faces, torusenes can be completely tessellated by hexagons. From a graph-theoretic viewpoint, a torusene is a cubic graph, embedded into the torus in such a way that each face is a hexagon.

It is well-known that symmetries in molecular graphs have a significant role in spectroscopy. For example, the graph-theoretic concepts of one-regularity and half-arc-transitivity have their chemistry counterpart in the concept of chirality. We note further that many chiral torusenes are also arc-transitive which are the ones in the focus of this IP's interest and what are the symmetry properties that could be used for obtaining relevant, chemically applicable geometric representations of such torusenes.

Special structures giving interesting (partial) geometries Very often discrete geometric structures, configurations, etc. are parallel realizations of abstract combinatorial structures reflecting a high level of symmetry, thus forming a natural link between algebraic graph theory and geometry. In this context permutation groups and coherent configurations form an algebraic background in algebraic graph theory and are related to diverse investigations in geometry, see e.g. [2]. A significant particular case of coherent configurations is given by association schemes [1]. Metric association schemes with d classes are canonically generated by distance regular graphs of diameter d, and distance regular graphs of diameter 2 are usually referred to as strongly regular graphs.

Our aim in research of the topic is to attack the problem of systematical enumeration of all partial geometries $pg(8, 9, 4)$. The solution should be based on an extensive use of computer aided facilities in conjunction with theory of coherent configuration. We expect to reach some progress along this line based on our experience with computer experimentation [5] and on fruitful earlier collaborations between members of the planed team, see e.g. [3, 12]. Success in solution will definitely be resulted in a creation of a new algorithmic background which will later on allow to investigate partial geometries more systematically.

1.2 Methodologies

In our research we will use a wide range of theories, results and tools of algebraic, geometric, combinatorial and topological nature, such as the theory of arc-transitive graphs with cyclic vertex stabilizers, embedding of graphs on surfaces, covering spaces, (quasi)primitive and genuinely imprimitive groups, solvable groups, finite geometries, computer algebra tools, group representations, group characters and coherent, to name just the most obvious. Also, we anticipate to use the well-known computational algebra systems, such as MAGMA and COCO.

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