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Maps, Symmetries, Configurations and Colourings

1 Individual Project's contribution to the CRP

1.1 Aims and Objectives

Our project is aimed at the interplay between algebraic, combinatorial, and geometric structures on graphs with focus on the relationships between graph embeddings, maps, and their symmetries on the one hand, and colourings, nowhere-zero flows, and geometrical configurations on the other hand.

Graph embeddings and immersions The concept of a graph embedding is central to our research. A 2-cell embedding of a graph G in a closed surface S is a homeomorphism $G \hookrightarrow S$ such that each component of $S-G$ is an open 2-cell. An immersion of a graph allows the images of two edges to intersect in some internal points, usually subject to certain natural restrictions. Although the study of graph embeddings and immersions forms the proper core of topological graph theory, motivations come from various areas of mathematics, physics, chemistry, and even computer science. Within the present project we will examine various aspects of graph embeddings – their structure, symmetries, extremal properties, various invariants, and others.

In topological graph theory, there are only a very few problems for which deterministic polynomial-time algorithms have been found. One of such problems is that of computation the maximum genus of a graph [21]. However, the latter result only presents a reduction of the problem to the matroid parity problem. It is one of our goals within this IP to design a new, direct, and simpler, algorithm for determining this invariant and to generalize it to embeddings of signed graphs. We also plan to explore embeddings that are close to maximum genus embeddings and retain certain features common with them.

While the classical approach to graph embeddings in surfaces requires the edges of the embedded graph not to intersect in their internal points, both practical applications and theoretical considerations lead to the study of immersions of graphs where internal intersections of edges are allowed. Particularly important is the case where the underlying surface is the plane and the number of edge intersections is minimal. This minimal number, called the crossing number of a graph, has received a significant attention in the literature, see [57, 58]. Crossing numbers of graphs are hard to compute: the precise numbers are known only for a few families of graphs. In our project we will investigate crossing numbers of several interesting families of graphs that may have a application in computer science and VLSI design.

Highly symmetric maps The theory of maps on surfaces focuses on two main aspects: symmetry and structure. Substantial results in both directions have led to deep implications to other branches of mathematics, such as hyperbolic geometry, group theory and theory of Riemann surfaces. An excellent survey of the connections was given by Jones and Singerman [35]. In this part of the Proposal we focus on symmetry, with particular emphasis on extremal richness in symmetries of a map.

Classification of regular maps is an important problem because of the outlined implications in group theory, hyperbolic geometry of tessellations, and Riemann surfaces. Despite considerable effort and a large number of contributions by both graph and group theorists, it is still a largely open area because of its enormous difficulty. Development in this field of research has been very dynamic over the past two decades, with a substantial contribution of investigators included in this project. For example, orientably regular embeddings of complete bipartite graphs have been recently classified by G. Jones (submitted), based on earlier results of the team members R. Nedela and M. Škoviera [17, 18, 31, 32]. The co-authorship of team members also led to a classification of regular embeddings of hypercubes [7]. R. Nedela and J. Širáň (with A. Breda) have given the first-ever classification of regular maps for an infinite family of surfaces, those of negative prime Euler characteristic [4]. In this Proposal we would like to explore possibilities for further results towards classification of regular maps with a given automorphism group.

Recent development of hardware and computer algebra systems such as MAGMA and GAP allows us to handle relatively large highly symmetrical structures including regular maps, vertex-transitive maps, and point-transitive geometric configurations. For instance, AP M. Conder in 2008 compiled a list of regular maps up to genus 202 (see his home page [8]). Our goal is to adopt selected known algorithms for computation in groups to compilation of lists of equivalence classes of "small" symmetrical structures of preassigned types, including their invariants. Such lists can then be applied to formulate, verify, or disprove conjectures about selected classes of objects. With the help of the structured information accompanying computer-generated lists one may then be able to extend the findings to formal proofs of existence of infinite families of objects with preassigned algebraic properties.

Colourings, flows, and configurations Graph colourings constitute the core of graph theory. While being essential in "pure" graph theory for understanding the structure of graphs, colourings are ubiquitous in the modelling of real world applications. Recently, they have been appearing in frequency assignments in telecommunications, in register allocations in the theory of operating systems, in routing and construction of interconnection networks, in optimization of WDM optical networks, and in many other areas.

Graph colouring problems are usually easy to state but exceedingly hard to solve. In fact, many known problems related to graph colourings are known to be NP-complete [23]. That makes them suitable for various algorithm testing, for example local search or genetic algorithms. Graph colouring problems come in a surprisingly great variety: vertex-colourings, edge-colourings, colourings with local restrictions, various types of graph homomorphisms – to mention just a few (for more detail see [37]). Several research teams have recently come up with new approaches to colourings; including circular colourings [61], oriented colourings [20], and colourings with Steiner triple systems [24, 26, 41, 45]. It is our intention to obtain further results in some of these areas.

The newly introduced concept of local Tait colourings of cubic graphs [24, 26, 41, 45] has opened surprising connections of colourings of cubic graphs to the combinatorial design theory and to finite geometry. A local Tait colouring is a generalization of a 3-edge-colouring of a cubic graph in which the global restriction on the number of colours is replaced by a local one. This can be done by allowing an arbitrary number of colours but requiring that any two colours meeting at a vertex uniquely determine the third colour. If we interpret the colours as points of some geometry, the previous requirement becomes identical to requiring that the three colours at any vertex of the graph lie on the same line of a suitable geometric configuration. This brings a new hierarchy into the edge-colourings with more than three colours. As indicated in the paper [41] co-authored by team members Máčajová and Škoviera, such colourings may shed a new light on some of the long-standing conjectures concerning snarks and unveil further interesting connections. It is and astounding fact that both the 5-Flow Conjecture and the Fulkerson Conjecture can be viewed as local Tait colouring problems where the respective point-line configurations are the the famous Desargues configuration and the Cremona-Richmond configuration known from projective geometry. We therefore intend to continue with this research.

1.2 Methodologies

The principal methods and tools in the course of the work on the project will be as follows:

- *covering spaces method* for constructing objects – graphs, embeddings, designs, flows, and others – with prescribed symmetry properties;
- *methods of combinatorial group theory* for the analysis of the symmetry groups of combinatorial objects;

- *geometric methods* for the study of maps as quotients of hyperbolic tessellations;
- *group representation methods* for handling the corresponding hyperbolic isometry groups;
- *spectral methods* in the investigation of issues related to construction and classification of strongly regular graphs;
- *methods of finite geometry* for the study of flows and colourings in graphs;
- *Pólya's method* and the *generating functions method* for enumeration of combinatorial structures.

The *method of computational experiments* has also proved extremely useful in a number of important cases, including discoveries of new infinite families of highly symmetric maps, graphs and other discrete structures.

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