Regular maps with simple underlying graphs

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Abstract

A regular map is a symmetric embedding of a graph (or multigraph) on some closed surface. In this paper we consider the genus spectrum for such maps on orientable surfaces, with simple underlying graph. It is known that for some positive integers g, there is no orientably-regular map of genus g for which both the map and its dual have simple underlying graph, and also that for some g, there is no such map (with simple underlying graph) that is reflexible. We show that for over 83% of all positive integers g, there exists at least one orientably-regular map of genus g with simple underlying graph, and conjecture that there exists at least one for every positive integer g.

1 Introduction

Regular maps are highly symmetric embeddings of graphs or multigraphs on closed surfaces. They generalise the Platonic solids (when these are viewed as embeddings of their 1-skeletons on the sphere) and the regular triangulations, quadrangulations and hexagonal tilings of the torus, to orientable surfaces of higher genus, and to non-orientable surfaces as well.

The formal study of regular maps was initiated by Brahana [2] in the 1920s and continued by Coxeter (see [8]) and others decades later. Deep connections exist between regular maps and other branches of mathematics, including hyperbolic geometry, Riemann surfaces and, rather surprisingly, number fields and Galois theory. See some of the references at the end of this paper for further background.

Regular maps on the sphere and the torus and other orientable surfaces of small genus are now quite well understood, but until recently, the situation for surfaces of higher genus was something of a mystery. A significant step towards answering some long-standing questions about the genera of orientable surfaces carrying a regular map having no multiple edges, or an 'orientably-regular' map that is chiral (admitting no reflectional symmetry) was taken by Conder, Siráň and Tucker in [7], after the first author noticed patterns in computational data about regular maps of small genus (see [3] and the associated lists of maps available on the first author's website).

One question of interest has been the genus spectrum of orientably-regular maps with simple underlying graph — that is, where the embedded graph has no loops or multiple edges. It is well known that for every $g > 0$ there exists a reflexible regular map of type $\{4g, 4g\}$ on an orientable surface of genus g (with dihedral automorphism group). It follows that there are no 'gaps' in the genus spectrum of orientable surfaces carrying reflexible regular maps. On the other hand, the underlying graphs for these maps are highly degenerate, being bouquets of $2g$ loops based at a single vertex.

A closely-related question concerns the genera of those orientably-regular maps with the property that the underlying graphs of both the map and its dual are simple. From the evidence described in [3], it was discovered that there are gaps in this spectrum: there are no such maps of genus 20, 23, 24, 30, 38, 39, 44, 47, 48, 54, 60, 67, 68, 77, 79, 80, 84, 86, 88 or 95, but there is at least one of genus g for every other g in the range $0 \leq g \leq 101$.

Two of the main results of [7] were that (a) If M is an orientably-regular but chiral map of genus $p+1$, where p is prime, and $p-1$ is not divisible by 5 or 8, then either M or its topological dual M^* has multiple edges, and (b) if M is a reflexible regular map of genus $p+1$, where p is prime and $p > 13$, then either M or M^* has multiple edges, and if also $p \equiv 1 \bmod 6$, then both M and M^* have multiple edges.

It follows from these that if $g = p + 1$ for some prime $p > 13$ such that $p - 1$ is not divisible by 5 or 8, then there exists no orientably-regular map of genus g such that the underlying graphs of both the map and its dual are simple. Hence there are infinitely many exceptions, well beyond the brief list given two paragraphs above.

On the other hand, if we are happy for just one of M and M^* to have simple underlying graph, then the situation is intriguing. The exceptions arising from (b) for reflexible regular maps are genera of the form $g = p+1$ where p is a prime congruent to 1 mod 6, but for each of these, there is a an orientably-regular but chiral map of type $\{6, 6\}$ of genus g with simple underlying graph. Hence these exceptions for reflexible maps are not exceptions for chiral maps.

In fact, it is easy to see from the Platonic maps, the toroidal regular maps and the lists of all regular maps of small genus (associated with [3]) that for every integer g in the range $0 \leq g \leq 101$, there exists at least one orientably-regular map of genus g with simple underlying graph.

Hence the obvious question arises: is there any positive integer g for which there exists no orientably-regular map of genus g with simple underlying graph?

We are prepared to conjecture that the answer is 'No', but a proof would be difficult. In this paper, we provide further evidence in support of it, by proving the existence of several infinite families of examples, covering various pieces of the genus spectrum.

We construct the maps via their automorphism groups (or at least their orientation-preserving groups of automorphisms), using a a range of combinatorial group-theoretic techniques. These include semi-direct product constructions (as used in [5] to produce regular maps on non-orientable surfaces of over 77% of all possible genera), and some more general methods similar to those used recently to find abelian regular covers of symmetric cubic graphs [6].

Some further background on regular maps is given in Section 2, and various infinite families are described in Section 3, to provide examples that cover over 83% of all possible genera. Many of the resulting gaps in the genus spectrum can be filled by maps of type $\{6, 6\}$, and some of these are described in Section 4, and then we give our main theorem and make some concluding remarks in Section 5.

Before proceeding, we note that very little of this work would have been likely without the benefit of the use of the computational algebra system MAGMA [1] to produce examples of small genus and to experiment with a number of constructions for infinite families.

2 Background on regular maps

A *map* is a 2-cell embedding of a connected graph X into a closed surface S without boundary. Note that the graph X can have loops and/or multiple edges. The term '2-cell' means that there are no edge-crossings, and each component (or *face*) of the complement $S \setminus X$ of the graph in the surface is simply connected – that is, homeomorphic to an open disk in $\mathbb{R}^2$.

The map M is called *orientable* or *non-orientable* according to whether the carrier surface is orientable or non-orientable, and the *genus* and the *Euler characteristic* of the map M are defined as the genus and the Euler characteristic of that surface. The topological *dual* of an orientable map M (which is denoted by M^*) is obtained from M by interchanging the roles of vertices and faces in the usual way.

Any such map M is composed of a vertex-set, an edge-set, and the set of its faces, denoted by $V = V(M)$, $E = E(M)$ and $F = F(M)$, respectively. The Euler characteristic χ of M is then given by the Euler-Poincaré formula $\chi = |V| - |E| + |F|$, and then the genus g of M is given by $\chi = 2 - 2g$ when M is orientable, or $\chi = 2 - g$ when M is non-orientable.

Associated also with any map M is a set of *darts*, or *arcs*, which are the incident vertex-edge pairs $(v, e) \in V \times E$; these can also be viewed as ordered pairs of adjacent vertices when the underlying graph is simple. Also each dart is associated with two *blades*, which consist of the dart (v, e) and a chosen side along the edge e; in the non-degenerate cases where every edge lies in two faces, these are the incident vertex-edge-face triples $(v, e, f) \in V \times E \times F$.

An *automorphism* of a map M is any permutation of the edges of the underlying graph that preserves incidence (and hence preserves the embedding), or equivalently, any automorphism of the graph induced by a homeomorphism of the carrier surface to itself. It is important to observe that by connectedness, *every automorphism of a map is uniquely determined by its effect on any blade.*

The set of all automorphisms of a map M forms a group under composition, called the *automorphism group* of a map, and denoted by $\mathrm{Aut}(M)$. If M is orientable, then the subgroup of all orientation-preserving automorphisms has index 1 or 2 in $\mathrm{Aut}(M)$, and is denoted by $\mathrm{Aut}^o(M)$, and sometimes also called the *rotation group* of M. If the orientable map M admits an orientation-reversing automorphism (so that $\mathrm{Aut}^o(M)$ has index 2 in $\mathrm{Aut}(M)$), then M is said to be *reflexible*, and otherwise M is *chiral*. On the other hand, if M is non-orientable, there is no such distinction.

A map M is called *orientably-regular* if it is orientable and $\mathrm{Aut}^o(M)$ acts regularly on the set of all darts of M. If such a map M is reflexible, then $\mathrm{Aut}(M)$ acts regularly on the set of all blades of M. Similarly, a non-orientable map M is called *regular* if $\mathrm{Aut}(M)$ acts regularly on the set of all blades of M. In general, a map is called *regular* if it is either orientably-regular or non-orientable and regular. Just to make it clear: regular maps fall into three classes: maps that are orientably-regular and reflexible, maps that are orientably-regular but chiral, and maps that are non-orientable and regular.

For any regular map M, the action of $\mathrm{Aut}(M)$ is transitive on the darts of M, and hence on the vertices, on the edges, and on the faces of M. It follows that every face of a regular map M has the same size, say m, and every vertex has valence, say k, and then M is said to have *type* $\{m, k\}$. The Platonic solids give the most famous examples, of types $\{3, 3\}$ (tetrahedron), $\{3, 4\}$ (octahedron), $\{4, 3\}$ (cube), $\{3, 5\}$ (icosahedron) and $\{5, 3\}$ (dodecahedron), while every regular map on the torus has type $\{3, 6\}$, $\{4, 4\}$ or $\{6, 3\}$. Note that if the regular map M has type $\{m, k\}$, then its dual (obtained by interchanging the roles of vertices and faces) has type $\{k, m\}$.

Now suppose M is a regular map of type $\{m, k\}$, let (v, e) be any dart of M, and let f be a face incident with e. Then by transitivity, there exists an automorphism R of M that preserves f and induces locally a single-step rotation about the centre of f, and this has order m. Similarly, there exists an automorphism S of M that fixes v and induces a single-step rotation around v, and this has order k. Moreover, we can choose each of r and s (either locally 'clockwise' or 'anti-clockwise') so that their product rs is an automorphism of order 2 that preserves e and acts locally like a rotation about the mid-point of e; in particular, r and s satisfy the relations $r^m = s^k = (rs)^2 = 1$. By connectedness, r and s generate a dart-transitive group of automorphisms of M, which must be either $\mathrm{Aut}(M)$ itself, or $G = \mathrm{Aut}^o(M)$ in the case where M is orientable and both r and s preserve orientation.

(The existence of such automorphisms is key to the definition of an alternative term for regular map, namely *rotary map*, as coined by Steve Wilson. This has the advantage of allowing the term 'regular' to be reserved for those rotary maps M with the property that $\mathrm{Aut}(M)$ acts regularly on blades, but recent usage has extended this term to cover the orientably-regular but chiral maps as well.)

It follows from the above observations that $\mathrm{Aut}(M)$ or $\mathrm{Aut}^o(M)$ is a quotient of the ordinary $(2, k, m)$ *triangle group*

$$\Delta^o(2, k, m) = \langle\, x, y \mid x^2 = y^k = (xy)^m = 1 \,\rangle,$$

under an epimorphism taking x to rs and y to s^{-1}. Note that the dual M^* is also regular, with the roles of r and s (and hence the roles of xy and y^{-1}) interchanged.

If M admits also an (involutory) automorphism a which reverses the edge e but (unlike rs) preserves each of the two blades associated with (v, e), then the action of $\mathrm{Aut}(M)$ is transitive on blades and so M is either reflexible or non-orientable. Also in this case $r^a = r^{-1}$ and $(rs)^a = (rs)^{-1} = rs$, and if we define $b = ar$ and $c = bs$, then a, b and c generate $\mathrm{Aut}(M)$ and satisfy the relations

$$a^2 = b^2 = c^2 = (ab)^m = (bc)^k = (ac)^2 = 1,$$

which are the defining relations for the full (or extended) triangle group $\Delta(2, k, m)$.

Note that the automorphism a is may be considered geometrically as a reflection, about an axis passing through the midpoints of the edge e and the face f. Similarly, the automorphisms b and c may be considered as a reflection about an axis through v and the midpoint of f (with $r^b = r^{-1}$ and $s^b = s^{-1}$) and a reflection about an axis through v and the midpoint of e (with $(rs)^c = (rs)^{-1} = rs$ and $s^c = s^{-1}$).

Conversely, given any epimorphism $\psi\colon \Delta^o \to G$ from the ordinary $(2, k, m)$ triangle group $\Delta^o = \Delta^o(2, k, m)$ onto a finite group G, in which the orders 2, k and m of the generators x, y and xy are preserved, a map M can be constructed using right cosets of the images of $\langle y\rangle$, $\langle x\rangle$ and $\langle xy\rangle$ as the vertices, edges and faces of M, respectively, with incidence given by non-empty intersection of cosets. (For example, the ordered pair $(v, e) = (\langle y^\psi\rangle, \langle x^\psi\rangle)$ is a dart of M, incident with the face $f = \langle (xy)^\psi\rangle$.) Also the group G acts naturally and transitively by right multiplication on each of $V(M)$, $E(M)$ and $F(M)$, preserving incidence, and transitively on the darts of M. It follows that M is a regular map of type $\{m, k\}$, with $G = \mathrm{Aut}^o(M)$ or $\mathrm{Aut}(M)$.

This map M admits also the automorphisms a, b and c (described above) if and only if the epimorphism ψ extends to an epimorphism $\widetilde{\psi} : \Delta \to \widetilde{G}$ from the full $(2, k, m)$ triangle group $\Delta = \Delta(2, k, m)$ onto a group $\widetilde{G}$ containing G as a subgroup of index 1 or 2. If G has index 2 in $\widetilde{G}$ then M is orientable and reflexible, while if $G = \widetilde{G}$ then M is non-orientable, and vice versa. In both cases, the kernel $K = \ker\psi$ is normal in Δ. On the other hand, if $K = \ker\psi$ is not normal in Δ, then M is orientable but chiral, and the conjugate of K by any element of $\Delta \setminus \Delta^o$ is the kernel of the epimorphism corresponding to the 'mirror image' of M.

In practice, we can tell whether or not an orientably-regular map M of type $\{m, k\}$ is reflexible, either by testing for an automorphism of $\mathrm{Aut}^o(M)$ that inverts the generating pair (r, s) (or the generating pair (rs, s)), or by testing whether the kernel K of the epimorphism $\psi\colon \Delta^o(2,k,m) \to \mathrm{Aut}^o(M)$ is invariant under conjugacy by an element of $\Delta \setminus \Delta^o$.

From this point of view, the study of regular maps can be reduced to the study of non-degenerate quotients of triangle groups.

As is well known (and shown in [7]), the simplicity of the underlying graphs can also be reduced to some easy group theory. If $\langle s\rangle$ stabilises the vertex v, then $\langle s^r\rangle$ stabilises the neighbouring vertex v^r, and their intersection stabilises both vertices. It follows that the existence of multiple edges between v and its neighbour v^{rs} is equivalent to the intersection $\langle s\rangle \cap \langle s^{rs}\rangle$ being non-trivial, and since the latter is normalised by both s and rs, it is normal in $\langle s, rs\rangle = \langle r, s\rangle = \mathrm{Aut}^o(M)$ or $\mathrm{Aut}(M)$.

Hence if M is an orientably-regular map, then M has simple underlying graph if and only if no non-trivial subgroup of the vertex-stabilser is normal in $\mathrm{Aut}^o(M)$.

Note that if $\mathrm{Aut}^o(M)$ is a non-abelian simple group, or 'almost simple' (or more generally, if every minimal normal subgroup of $\mathrm{Aut}^o(M)$ is a non-abelian simple group), then every cyclic subgroup must be core-free in $\mathrm{Aut}^o(M)$, and in that case, both M and M^* have simple underlying graph. The genera of such maps, however, are somewhat sparse, and so this observation is of little use to us. In the next section, we produce families of covers of known examples, with the automorphism group each cover having a cyclic normal subgroup of fixed index.

3 Construction of helpful families of maps

In this Section, we construct several families of orientably-regular maps with simple underlying graph, which will help us prove our main theorem.

3.1 Family A: Orientably-regular maps of type $\{3n, 4\}$

It is well known that a regular octahedron can be viewed as a regular embedding of its 1-skeleton (which is a 4-valent graph of order 8) on the sphere, giving a Platonic regular map M, of type $\{3, 4\}$ and genus 0. The automorphism group of this map is $S_4 \times C_2$, with the S_4 preserving orientation.

It is also well-known that an infinite family of regular maps of type $\{3n, 4\}$ can be constructed as cyclic regular coverings of the octahedral map; see [10] or [13] for example. These maps can be constructed in a number of ways.

One way is by using semi-direct products, in a way similar to the approach taken in [5]: for any positive integer n, form the semi-direct product $G = NH \cong C_{3n} \rtimes S_4$ of a cyclic group $N = \langle\, w \mid w^{3n} = 1 \,\rangle$ of order $3n$ by the symmetric group $H = S_4 = \langle\, u, v \mid u^2 = v^4 = (uv)^3 = 1 \,\rangle$, with conjugation of N by H given by $w^u = w^{-1}$ and $w^v = w^{-1}$. In this group, define $x = wu$ and $y = v$; then $x^2 = y^4 = 1$ and $(xy)^3 = (wuv)^3 = w^3$, which has order n, so the subgroup of G generated by x and y has order $24n$ and is the rotation group of an orientably-regular map of characteristic $\chi = 24n/(3n) - 24n/2 + 24n/4 = 8 - 6n$ and genus $g = 3n - 3$.

The vertex-stabiliser (in the rotation group $\langle x, y\rangle$) is the cyclic subgroup of order 4 generated by $y = v$, and as this contains no non-trivial normal subgroup of $\langle x, y\rangle$, the underlying graph of the map is simple. (On the other hand, the face-stabiliser is the cyclic subgroup of order $3n$ generated by $xy = wuv$, which contains the cyclic normal subgroup generated by $(xy)^3 = w^3$, and so the underlying graph of the dual map has multiple edges, for $n > 1$.) Also each map is reflexible, since the involutory automorphism of $H = S_4$ inverting each of u and v extends to an involutory automorphism of G that inverts each of $x = wu$ and y (and centralises w).

Another way to construct these maps is to take the group Γ with presentation $\Gamma = \langle\, x, y \mid x^2 = y^4 = (xyxy^2)^2 = 1\,\rangle$, and consider the normal subgroup N of index 24 in Γ generated by the element $z = (xy)^3$. Now the relation $(xyxy^2)^2 = 1$ can be re-written as $(yxyxy)^2 = 1$, which gives $(yx)^3 = (yxyxy)x = (y^{-1}xy^{-1}xy^{-1})x = (xy)^{-3}$, and it follows that conjugation by each of x and y^{-1} inverts the element $z = (xy)^3$. Hence in particular, z generates a cyclic normal subgroup K in Γ, of index 24, with quotient $\Gamma/K = \langle\, x, y \mid x^2 = y^4 = (xyxy^2)^2 = (xy)^3 = 1\,\rangle \cong S_4$. Next, by Reidemeister-Schreier theory (see [9] or [12]), or by use of the `Rewrite` command in Magma [1], we find that the subgroup K is free of rank 1, and hence infinite. Thus for each positive integer n, we can factor out the normal subgroup generated by z^n, and get an extension of C_n by S_4, just as above. The resulting orientably-regular map has type $\{3n, 4\}$ and genus $g = 3n - 3$, and is reflexible (because its rotation group admits an involutory automorphism that inverts the images of the two generators x and y), and its underlying graph is simple, because the cyclic subgroup of order 2 generated by the image of y^2 is not normal in the rotation group.

A presentation for the rotation group of the nth map in this family is simply

$$\langle\, r, s \mid (rs)^2 = s^4 = (r^2s^{-1})^2 = r^{3n} = 1\,\rangle,$$

which can be obtained by taking $r = xy$ and $s = y^{-1}$. Similarly, a presentation for the full automorphism group is

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (bc)^4 = (ababcb)^2 = (ab)^{3n} = 1\,\rangle,$$

with the rotation group generated by $r = ab$ and $s = bc$ as usual. The first few members of this family (after the first one, of genus 0) are the duals of the maps named R3.4, R6.3, R9.11, R12.1 and R15.5 in [4].

For the purposes of this paper, the important point is summarised in the following:

Proposition 3.1 *For every positive integer n, there exists a reflexible regular map of type $\{3n, 4\}$ and genus $3n - 3$, with simple underlying graph (and rotation group an extension of C_n by S_4).*

3.2 Family B: Orientably-regular maps of type $\{4n, 4\}$

In this case we can start with the toroidal map of type $\{4, 4\}_4$ (see [8]), with rotation group $H = \langle\, u, v \mid u^2 = v^4 = (uv)^4 = [u, v]^2 = 1\,\rangle$, which is an extension of $C_2 \times C_2$ by D_4, of order 32. and construct an infinite family of cyclic regular coverings of this.

As in the previous case, for any positive integer n we can form the semi-direct product $G = NH \cong C_{4n} \rtimes H$ of a cyclic group $N = \langle\, w \mid w^{4n} = 1 \,\rangle$ of order $4n$ by the above group H, with conjugation of N by H given by $w^u = w^{-1}$ and $w^v = w^{-1}$. Again we may define $x = wu$ and $y = v$ in his group, and this time we find $x^2 = y^4 = 1$ while $(xy)^4 = (wuv)^4 = w^4$, which has order n, so the subgroup of G generated by x and y has order $32n$ and is the rotation group of an orientably-regular map of characteristic $\chi = 32n/(4n) - 32n/2 + 32n/4 = 8 - 8n$ and genus $g = 4n - 3$.

Again the vertex-stabiliser $\langle y \rangle$ (of order 4) contains no non-trivial normal subgroup of $\langle x, y \rangle$, so the underlying graph of the map is simple, while the face-stabiliser $\langle xy \rangle$ (of order $4n$) contains the cyclic normal subgroup generated by $(xy)^4 = w^4$, and so the underlying graph of the dual map has multiple edges, for $n > 1$. Also each map is reflexible, since the automorphism of H inverting each of u and v extends to an involutory automorphism of G that inverts each of $x = wu$ and y (and centralises w).

For an alternative construction, take the normal subgroup N of index 32 generated by $z = (xy)^4$ in the group $\Phi = \langle\, x, y \mid x^2 = y^4 = xyxy^2xy^{-1}xy^2 = 1 \,\rangle$. In this group, the relation $xyxy^2xy^{-1}xy^2 = 1$ can be re-written as $yxyxy = y^{-1}xyxy^{-1}$, giving

$$\begin{aligned}(yx)^4 &= yx(yxyxy)x = yx(y^{-1}xyxy^{-1})x = (yxy^{-1}xy)xy^{-1}x = (y^{-1}xyxy^{-1})^{-1}xy^{-1}x \\ &= (yxyxy)^{-1}xy^{-1}x = (y^{-1}xy^{-1}xy^{-1})xy^{-1}x = (y^{-1}x)^4 = (xy)^{-4},\end{aligned}$$

from which it follows that $z = (xy)^4$ is inverted under conjugation by x and y. Accordingly, z generates a cyclic normal subgroup K of index 32 in Φ, with quotient $\Phi/K = \langle\, x, y \mid x^2 = y^4 = xyxy^2xy^{-1}xy^2 = (xy)^4 = 1 \,\rangle \cong H$, and by Reidemeister-Schreier we find that K is infinite.

Again for each positive integer n, we can factor out the normal subgroup generated by z^n, and get an extension of C_n by H. The resulting orientably-regular map has type $\{4n, 4\}$ and genus $g = 4n-3$, and is reflexible (since $\langle x, y \rangle$ admits an involutory automorphism that inverts x and y and centralises $z = (xy)^4$), and its underlying graph is simple (since $\langle y^2 \rangle$ is not normal in the rotation group).

In fact the group we obtain in this way has the same presentation (in terms of the images of x and y) as the group defined using the semi-direct product construction, since in the former case, the relations $uw = w^{-1}u$ and $vw = w^{-1}v$ imply that

$$xyxy^2xy^{-1}xy^2 = wuvwuv^2wuv^{-1}wuv^2 = w^{1+1-1-1}uvuv^2uv^{-1}uv^2 = uvuv^2uv^{-1}uv^2,$$

which is trivial. We will exploit this fact in the next sub-section.

Meanwhile we have the following:

Proposition 3.2 *For every positive integer n, there exists a reflexible regular map of type $\{4n, 4\}$ and genus $4n - 3$, with simple underlying graph (and rotation group an extension of C_n by the rotation group of the toroidal map of type $\{4, 4\}_4$).*

A presentation for the rotation group of the nth map in this family is simply

$$\langle\, r, s \mid (rs)^2 = s^4 = (rs^{-1})^2(r^{-1}s)^2 = r^{4n} = 1 \,\rangle,$$

which again can be obtained by taking $r = xy$ and $s = y^{-1}$.

Note that the relation $xyxy^2xy^{-1}xy^2 = 1$ can be rewritten as $1 = xy^2xy^{-1}xy^2xy$, and when we take $r = xy$ and $s = y^{-1}$, this gives $1 = rs^{-1}rs^{-2}rs^{-1}r = (rs^{-1})^2(s^{-1}r)^2$. On the other hand, it can also be rewritten as $1 = xy^2xyxy^2xy^{-1} = xy^2xyxy^{-2}xy^{-1} = (xy^2)^2(y^{-1}xy^{-1})^2$, which becomes $1 = (rs^{-1})^2(r^{-1}s)^2$. Hence we find that $(r^{-1}s)^2 = (rs^{-1})^2 = (s^{-1}r)^2$, and in particular, $(rs^{-1})^4 = (r^{-1}s)^2(r^{-1}s)^2 = (r^{-1}s)^2(s^{-1}r)^2 = 1$. Again we will use this in the next sub-section.

A presentation for the full automorphism group is

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (bc)^4 = (abcb)^2(babc)^2 = (ab)^{4n} = 1 \,\rangle,$$

with the rotation group generated by $r = ab$ and $s = bc$ as usual. The first few members of the resulting family (after the first one, of genus 1) are the duals of the maps named R5.6, R9.10 and R13.4 in [4].

3.3 Family C: Orientably-regular maps of type $\{8n, 4\}$

Take the group $\Phi = \langle\, x, y \mid x^2 = y^4 = xyxy^2xy^{-1}xy^2 = 1 \,\rangle$ from the previous case, and for any positive integer n, factor out the cyclic normal subgroup K generated by $z^{2n} = (xy)^{8n}$. The resulting quotient is the same as the one obtained using the semi-direct product construction, and in the group $G \cong C_{8n} \rtimes H$ from that, we have

$$[x, y^2] = xy^2xy^2 = wuv^2wuv^2 = w^{1-1}uv^2uv^2 = (uv^2)^2,$$

which is an involution that centralises every power of w. Since the element $(uv^2)^2$ is central in $H = \langle u, v\rangle$, this involution is centralised also by each of x $(= wu)$ and y $(= v)$, and therefore central in G. Also the element $(xy)^{4n}$ is the unique involution in the cyclic normal subgroup generated by $(xy)^4$, and hence is central in G as well.

These two central involutions $[x, y^2] = (uv^2)^2$ and $(xy)^{4n}$ are distinct, so their product $[x, y^2](xy)^{4n}$ is a third central involution. Taking the quotient of $\langle x, y\rangle$ of the central subgroup of order 2 generated by this third involution, we obtain a group of order $32(2n)/2 = 32n$, generated by two elements of orders 2 and 4 with product of order $8n$. This gives an orientably-regular map of type $\{8n, 4\}$, with characteristic $\chi = 32n/(8n) - 32n/2 + 32n/4 = 4 - 8n$, and genus $g = 4n - 1$.

Again the map is reflexible, since the automorphism inverting the two generators of the earlier group centralises both $[x, y^2]$ and $(xy)^{4n}$, and therefore centralises their product as well. Also the underlying graph of the map is simple (for the same reasons as before). Thus we have the following:

Proposition 3.3 *For every positive integer n, there exists a reflexible regular map of type $\{8n, 4\}$ and genus $4n - 1$, with simple underlying graph.*

A presentation for the rotation group of the nth map in this family is simply

$$\langle\, r, s \mid (rs)^2 = s^4 = (rs^{-1})^2(r^{-1}s)^2 = sr^{-1}sr^{-1+4n} = r^{8n} = 1 \,\rangle,$$

although the last relation is redundant, since the fourth relation $sr^{-1}sr^{-1+4n} = 1$ gives $r^{4n} = (rs^{-1})^2$ and therefore $r^{8n} = (r^{4n})^2 = (rs^{-1})^4 = 1$, by what we observed in the previous sub-section. A presentation for the full automorphism group is

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (bc)^4 = (abcb)^2(babc)^2 = bcbabc(ab)^{-1+4n} = 1\,\rangle,$$

with the rotation group generated by $r = ab$ and $s = bc$ as usual.

The first few members of the resulting family are the duals of the maps named R3.5, R7.3, R11.2 and R15.6 in [4]. Also the carrier surfaces of these maps can easily be shown to be the same as those considered by Kulkarni in [11]. On the other hand, the regular maps with the same parameters obtainable from Maclachlan's surface actions in [13] (which include the maps named R3.6, R7.4, R11.3 and R15.7 in [4]) do not have simple underlying graphs.

3.4 Family D: Orientably-regular maps of type $\{6n, 6\}$

In this sub-section we exhibit three families of orientably-regular maps of type $\{6n, 6\}$ and genus $6n - 2$, for $n \equiv 0, 1$ and 2 mod 3 respectively. The maps in the first family are chiral, while those in the second and third families are reflexible.

We begin with the second and third families. For $n \not\equiv 0$ mod 3, we take the group Σ with presentation

$$\Sigma = \langle\, x, y \mid x^2 = y^6 = xyxy^{-2}xy^{-1}xy^2 = 1\,\rangle,$$

and consider the two elements $u = (xy)^2$ and $v = (xy^2)^2$.

Note that the relation $xyxy^{-2}xy^{-1}xy^2 = 1$ is equivalent (by inversion and conjugation) to $xyxy^2xy^{-1}xy^{-2} = 1$. Also this relation implies that $xyx = y^{-2}xyxy^2$, and hence that $(yx)^2 = y(xyx) = y^{-1}xyxy^2 = (y^{-1}xy^{-1}x)(xy^2xy^2) = u^{-1}v$. Similarly, the relation $xyxy^{-2}xy^{-1}xy^2 = 1$ gives $xy^2x = y^{-1}xy^2xy$, and so $(y^2x)^2 = y^2(xy^2x) = yxy^2xy = y(y^{-1}xy^2xy)y = xy^2xy^2 = v$. From these observations, we deduce that

$$\begin{aligned} u^x &= (yx)^2 = u^{-1}v, & u^y &= y^{-1}xyxy^2 = u^{-1}v \text{ (as above)},\\ v^x &= (y^2x)^2 = xy^2xy^2 = v, & v^y &= y^{-1}xy^2xy^3 = (y^{-1}xy^2xy)y^2 = xy^2xy^2 = v. \end{aligned}$$

In particular, the subgroup N generated by u and v is normal in Σ. The quotient Σ/N is generated by the (involutory) images of the elements x and xy, and hence is dihedral of order 12, so N has index 12 in Σ. Also v is centralised by both generators of Σ, and hence by u, and therefore N is abelian. Moreover, $u^3 = (xy)^6$, while

$$\begin{aligned} v^3 &= (xy^2)^2(xy^2)^2(xy^2)^2 = (xy^2)^2(y^2x)^2(xy^2)^2 = xy^2xy^{-2}xy^{-2}xy^2\\ &= y^{-1}xy^2xyy^{-2}xy^{-2}xy^2 = y^{-1}(xy^2xy^{-1}xy^{-2}xy)y = 1. \end{aligned}$$

Thus Σ is isomorphic to an extension of $\mathbb{Z} \oplus Z_3$ by D_6. Also $(u^3)^x = (u^3)^y = (u^{-1}v)^3 = u^{-3}v^3 = u^{-3}$, and therefore the element u^3 generates a cyclic normal subgroup of Σ, with index 9 in N and index 108 in Σ. By Reidemeister-Schreier theory, this subgroup is infinite. (Some of these things can also be verified with the help of Magma.)

Now for any positive integer n, we may factor out the normal subgroup generated by u^{3n}, and get a quotient of order $108n$ that is the rotation group of an orientably-regular map of type $\{6n, 6\}$, characteristic $108n/(6n) - 108n/2 + 108n/6 = 18 - 36n$, and genus $18n-8$. This map is reflexible, since the group Σ admits an automorphism θ which inverts x and y, and takes $u = (xy)^2$ to $(xy^{-1})^2 = x(y^{-1}x)^2x = (u^{-1})^x = (u^{-1}v)^{-1} = uv^{-1}$ and $v = (xy^2)^2$ to $(xy^{-2})^2 = x(y^{-2}x)^2x = (v^{-1})^x = v^{-1}$, and this automorphism takes u^3 to $(uv^{-1})^3 = u^3$, so preserves the quotient $\Sigma/\langle u^{3m}\rangle$.

But also the normal subgroup $N/\langle u^{3n}\rangle$ of this quotient has a characteristic abelian subgroup of order 9 generated by $u^n = (xy)^{2n}$ and v (each of order 3).

If $n \equiv 1 \bmod 3$, say $n = 3d-2$, then the element $u^n v$ generates a cyclic normal subgroup of Σ, since $(u^n v)^y = u^n v$ and

$$(u^n v)^x = (u^{-1}v)^n v = u^{-n}v^{n+1} = (u^n v)^{-1}v^{n+2} = (u^n v)^{-1}v^{3d} = (u^n v)^{-1}.$$

When we factor out this subgroup of order 3, we get a quotient of order $36n$ that is the rotation group of an orientably-regular map of type $\{6n, 6\}$, characteristic $36n/(6n) - 36n/2 + 36n/6 = 6 - 12n$, and genus $6n-2$. The underlying graph of the map is simple for all such $n > 1$, since neither $\langle y^2\rangle$ nor $\langle y^3\rangle$ is normal in the rotation group, but that does not happen for the case $n = 1$ (since in that case $v = (xy^2)^2$ becomes trivial, so $(y^2)^x = y^{-2}$, which then makes $\langle y^2\rangle$ normal in $\langle x, y\rangle$). Also the map is reflexible (for all n), since the inverting automorphism θ of Σ takes $u^n v$ to $(uv^{-1})^n v^{-1} = u^n v^{-n-1} = (u^n v)v^{-n-2} = (u^n v)v^{-3d} = u^n v$.

Similarly, if $n \equiv 2 \bmod 3$, say $n = 3d+2$, then the element $u^n v^{-1}$ generates a cyclic normal subgroup of Σ, since $(u^n v^{-1})^y = u^n v^{-1}$ and

$$(u^n v^{-1})^x = (u^{-1}v)^n v^{-1} = u^{-n}v^{n-1} = (u^n v^{-1})^{-1}v^{n-2} = (u^n v^{-1})^{-1}v^{3d} = (u^n v^{-1})^{-1},$$

and when we factor out this subgroup of order 3, again we get a quotient of order $36n$ that is the rotation group of an orientably-regular map of type $\{6n, 6\}$, characteristic $36n/(6n) - 36n/2 + 36n/6 = 6 - 12n$, and genus $6n-2$, with simple underlying graph. Again the map is reflexible, since the inverting automorphism θ of Σ takes $u^n v^{-1}$ to $(uv^{-1})^n v = u^n v^{-n+1} = (u^n v^{-1})v^{-n+2} = (u^n v^{-1})v^{-3d} = u^n v^{-1}$.

Thus we get two families of reflexible maps with the desired properties. The first few members of these two families are the duals of the maps named R10.16, R22.9, R28.21, R40.5, R40.5 and R46.23 in the first author's website of orientably-regular maps of genus 2 to 101 (see [3]).

For given $n \not\equiv 0 \bmod 3$, a presentation for the rotation group of the map is given by

$$\langle\, r, s \mid (rs)^2 = s^6 = r^2s^3rs^2rs^{-1} = r^{2n}(rs^{-3})^{\pm 2} = 1 \,\rangle,$$

with the final superscript in the last relator being $+2$ for all $n \equiv 1 \bmod 3$, and -2 for all $n \equiv 2 \bmod 3$. Correspondingly, a presentation for the full automorphism group is

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (bc)^6 = (ab)^2(bc)^3acbcabcb = (ab)^{2n}(abcbcbcb)^{\pm 2} = 1 \,\rangle,$$

with the rotation group generated by $r = ab$ and $s = bc$ as usual.

For the case $n \equiv 0 \bmod 3$, the above approach does not work; indeed there is no such map (with the above parameters and with simple underlying graph) when $n = 3, 6, 9, 12$ or 15, for example. Instead, we start with a different group Λ, with presentation

$$\Lambda = \langle\, x, y \mid x^2 = y^6 = (xyxy^2)^2 = (xy^2)^2(xy^{-2})^2 = 1 \,\rangle.$$

The relations in this group imply $yxy^2xy = xy^{-2}x$, and hence $1 = xy^2xy^2xy^{-2}xy^{-2}$ $= xy(yxy^2xy)y^3xy^{-2} = xy(xy^{-2}x)y^3xy^{-2}$, which gives us $xy^{-2}xy^3 = y^{-1}xy^2x$.

Now take $u = xy^{-2}xy^2$ and $v = (xy)^3(xy^{-1})^3$. Then we find $u^x = y^{-2}xy^2x = u^{-1}$ and $v^x = (yx)^3(y^{-1}x)^3 = v^{-1}$, while $u^y = y^{-1}xy^{-2}xy^3 = y^{-1}y^{-1}xy^2x = u^{-1}$, and $v^{y^{-1}} = yvy^{-1} = (yx)^3(y^{-1}x)^3 = v^{-1}$, from which it follows that also $v^y = v^{-1}$. Hence both u and v are inverted under conjugation by each of x and y, and in particular, the subgroup N generated by u and v is normal in Λ. Moreover, since v is centralised by y^2, we find that v is centralised by $xy^{-2}xy^2 = u$, and so N is abelian.

Also $(xy)^6 = (xy)^3(xy^{-1})^3(yx)^3(xy)^3 = vyxyxy^2xyxy = vyx(xy^{-2}x)xy = v$. It follows from this (with the help of Magma [1] if necessary) that the quotient Λ/N has order 36, and then by Reidemeister-Schreier, we find that N is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}$, with the $\mathbb{Z}_3$ generated by u, and with v infinite. (Indeed

$u^3 = xy^{-2}xy^2xy^{-2}xy^2xy^{-2}xy^2 = xy^{-2}xy^2(y^2xy^{-2}x)xy^{-2}xy^2 = xy^{-2}xy^{-2}xy^2xy^2 = 1.$)

It follows that the subgroup generated by v itself is normal in N, with index 108, and when we factor out the subgroup generated by v^n for any positive integer n, we get a quotient of order $108n$ in which the image of xy has order $6n$. Accordingly, this quotient is again the rotation group of an orientably-regular map of type $\{6n, 6\}$, characteristic $108n/(6n) - 108n/2 + 108n/6 = 18 - 36n$, and genus $18n - 8$. Also this map is reflexible, since the rotation group admits an automorphism that inverts the images of x and y, and then takes the image of $u = xy^{-2}xy^2$ to the image of $xy^2xy^{-2} = y^2x(xy^{-2}xy^2)xy^{-2} = u^{xy^{-2}} = u^{-1}$ and similarly the image of $v = (xy)^3(xy^{-1})^3$ to the image of $(xy^{-1})^3(xy)^3 = (y^{-1}x)^3((xy)^3(xy^{-1})^3)(xy)^3 = v^{(xy)^3} = v$.

But if n is divisible by 3, say $n = 3m$, and we factor out the normal subgroup generated by uv^m, then we get a different quotient, of order $108m = 36n$, in which the image of xy has order $18m$, since the image of $v = (xy)^6$ has order $3m$ (with the image of v^m coinciding with the image of u^{-1}). This gives an orientably-regular map of type $\{18m, 6\}$, characteristic $108m/(18m) - 108m/2 + 108m/6 = 6 - 36m$, and genus $18m - 2 = 6n - 2$, but the map is no longer reflexible, since any automorphism that inverts the images of x and y must take the image of uv^m (which is trivial) to $u^{-1}v^m$ (which is not). On the other hand, the underlying graph is simple, since the images of the subgroups generated by y^2 and y^3 are not normal.

Thus we have a family of chiral maps of type $\{6n, 6\}$ and genus $6n - 2$, for n divisible by 3, all with simple underlying graphs.

A presentation for the rotation group (in the case $n = 3m$) is

$$\langle\, r, s \mid (rs)^2 = s^6 = (r^2s^{-1})^2 = (rs^{-1})^2(rs^3)^2 = rs^3rs^{-1}r^{n/3} = 1 \,\rangle,$$

again obtainable by taking $r = xy$ and $s = y^{-1}$.

The first few members of the resulting family are the duals of the maps named C16.1, C34.1, C52.1, C70.1 and C88.1 in the first author's website of orientably-regular maps of genus 2 to 101 (see reference [3]).

Thus we have the following:

Proposition 3.4 *For every integer $n > 1$, there exists an orientably-regular map of type $\{6n, 6\}$ and genus $6n-2$, with simple underlying graph. In fact if $n \not\equiv 0$ mod 3, then there exists such a map that is reflexible, while if $n \equiv 0$ mod 3, there exists such a map that is chiral.*

Note that the four families of maps we have described so far have genera congruent to 0, 1, 3, 4, 5, 6, 7, 9, 10 and 11 mod 12. For the remaining two congruence classes (namely 2 and 8 mod 12), maps described in the next section will be helpful.

4 Orientably-regular maps of type $\{6, 6\}$

A well-known family of regular maps of type $\{6, 6\}$ was introduced by Sherk [14] in the 1960s, using a construction based on the automorphism groups of the toroidal maps of type $\{3, 6\}$. The maps in Sherk's family are indexed by ordered pairs (α, β) of non-negative integers, with rotation group of the form

$$G_{\alpha,\beta} = \langle\, r, s \mid (rs)^2 = r^6 = s^6 = (r^2s^{-1})^2 = (r^{-2}s^{-2})^\alpha(r^2s^2)^\beta = 1 \,\rangle$$

for each such pair $(\alpha, \beta) \neq (0, 0)$. This group has order $12k$ where $k = \alpha^2 + \alpha\beta + \beta^2$, and the corresponding map (which we will denote by $\mathcal{S}_{(\alpha,\beta)}$) has genus $k+1$, and the map $\mathcal{S}_{(\alpha,\beta)}$ is reflexible if and only if $\alpha\beta(\alpha-\beta) = 0$. For example, $\mathcal{S}_{(0,1)}$, $\mathcal{S}_{(1,0)}$, $\mathcal{S}_{(0,2)}$, $\mathcal{S}_{(1,1)}$, $\mathcal{S}_{(2,0)}$, $\mathcal{S}_{(1,2)}$ and $\mathcal{S}_{(2,1)}$ are respectively the maps R2.5, R2.5, R5.10, the dual of R4.8, R5.10, the dual of C8.1, and the mirror image of the dual of C8.1 in [4].

The underlying graph of $\mathcal{S}_{(\alpha,\beta)}$ is simple except when $\alpha+\beta \leq 2$, for in those cases $\langle s^3\rangle$ or $\langle s^2\rangle$ is normal in $G_{\alpha,\beta}$ (while no such degeneracy occurs when $\alpha+\beta > 2$).

Hence the Sherk family gives orientably-regular maps of genus g (and type $\{6, 6\}$) with simple underlying graphs, for all g expressible in the form $\alpha^2+\alpha\beta+\beta^2+1$ where α and β are non-negative integers with $\alpha+\beta > 2$. This set of possible genera is 'quadratic' rather than 'linear', and so asymptotically is less dense than the arithmetic progressions of genera provided by the families in the sub-sections above, but nevertheless it covers some genera that the previous families do not, such as 8, 14, 20, 26, 32, 38, 44, 50, 62, 68, 74, 80, 92 and 98 (but not 56 or 86).

On the other hand, the underlying graph of the dual of $\mathcal{S}_{(\alpha,\beta)}$ is never simple, for the relation $(r^2s^{-1})^2 = 1$ can be rewritten as $(r^3(rs)^{-1})^2 = 1$, which implies that conjugation by the involution rs inverts r^3, and hence $\langle r^3\rangle$ is always normal in $G_{\alpha,\beta}$.

Below we will show that there exist other families of orientably-regular maps of type $\{6, 6\}$ with simple underlying graph that not only have genera covering some of the remaining gaps in the genus spectrum, but also have a dual with simple underlying graph as well. These will be obtained as covers of a particular map of type $\{6, 6\}$ and genus 2, namely the map R2.5 in [4], which has rotation group $C_2 \times C_6$.

One way is via a semi-direct product construction, similar to the one used before. Let n be any odd positive integer with the property that some unit t in $\mathbb{Z}_n$ has multiplicative order 6, and form the semi-direct product $G = NH$ of a cyclic group $N = \langle\, w \mid w^n = 1 \,\rangle$ of order n by the group $H = C_2 \times C_6 = \langle\, u, v \mid u^2 = v^6 = [u, v] = 1 \,\rangle$, with conjugation of N by H given by $w^u = w^{-1}$ and $w^v = w^t$. In this group, define $x = u$ and $y = wv$. Then $xyxy^{-1} = uwvuv^{-1}w^{-1} = w^{-2}uvuv^{-1} = w^{-2}$, which generates N, and it follows that x and y generate G. Clearly x has order 2, while the orders of $y = wv$ and $xy = uvw = w^{-1}uv$ are multiples of 6.

In fact, $y^6 = (wv)^6 = w^{1+t+t^2+t^3+t^4+t^5}$ and $(xy)^6 = (uwv)^6 = w^{-(1-t+t^2-t^3+t^4-t^5)}$, so y and xy have order 6 precisely when $1+t+t^2+t^3+t^4+t^5$ and $1-t+t^2-t^3+t^4-t^5$ are both congruent to 0 mod n. If these two conditions hold, then subtracting one from the other gives $2(1 + t^2 + t^4) \equiv 0 \bmod n$, and hence $1 + t^2 + t^4 \equiv 0 \bmod n$. Conversely, if $1+t^2+t^4 \equiv 0 \bmod n$, then $1+t+t^2+t^3+t^4+t^5 = (1+t)(1+t^2+t^4) \equiv 0 \bmod n$ and $1 - t + t^2 - t^3 + t^4 - t^5 = (1 - t)(1 + t^2 + t^4) \equiv 0 \bmod n$, so this group G can be constructed precisely when $1 + t^2 + t^4 \equiv 0 \bmod n$.

In that case, G is the rotation group of an orientably-regular map of type $\{6, 6\}$, characteristic $\chi = 12n/6 - 12n/2 + 12n/6 = -2n$, and genus $n + 1$. Moreover, it is easy to verify that $xy^{-2}xy^2 = w^{2(t+t^2)}$ and $xy^{-3}xy^3 = w^{2(t+t^2+t^3)}$, and it follows that the underlying graph of this map is simple if and only if $t + t^2$ and $t + t^2 + t^3$ are non-zero mod n. Similarly, $xy^{-2}xy^2 = w^{2(t-t^2)}$ and $xy^{-3}xy^3 = w^{2(t-t^2+t^3)}$, so the underlying graph of the dual map is simple if and only if $t - t^2$ and $t - t^2 + t^3$ are non-zero mod n. Also if the map has simple underlying graph then it must be chiral, since any automorphism of G that inverts each of x and y must conjugate $w^{-2} = xyxy^{-1}$ to $xy^{-1}xy = w^{-2}$, and therefore centralises w, but on the other hand, it must also conjugate $w^{2(t+t^2)} = xy^{-2}xy^2$ to $xy^2xy^{-2} = w^{-2(1+t^5)}$, so that $t + t^2 \equiv -(1 + t^5) = -(t^2 + t)t^5 \bmod n$, which is impossible.

Examples include some of the Sherk maps (of genus 8, 14, 20, 32, 38, 44, 50, 62, 68, 74, 80, 92 and 98 for example), but also others for which both the map and its dual have simple underlying graph, such as the maps C22.2, C40.2, C58.2, C92.1 and C94.2 from [3], arising when $(n, t) = (21, 10)$, $(39, 4)$, $(57, 46)$, $(91, 30)$ and $(93, 37)$.

Some other classes can be constructed as follows:

Let Ψ be the group with presentation $\Psi = \langle\, x, y \mid x^2 = y^6 = (xy)^6 = 1 \,\rangle$.

Then the derived group Ψ' of Ψ (which is generated by the conjugates of the element $[x, y]$) has index 12 in Ψ, with quotient Ψ/Ψ' isomorphic to $C_2 \times C_6$, which is the rotation group of the regular map R2.5. In fact, by Reidemeister-Schreier (or by using the `Rewrite` command in Magma), we find that the subgroup Ψ' is generated by the four elements

$$w_1 = xy^{-1}xy, \quad w_2 = xyxy^{-1}, \quad w_3 = xy^{-2}xy^2, \quad w_4 = xy^2xy^{-2},$$

subject to a single defining relation $w_2^{-1}w_4w_3^{-1}w_1w_2w_4^{-1}w_3w_2^{-1} = 1$. Note that the fifth commutator of the form $xy^{-i}xy^i$, namely $w_5 = xy^{-3}xy^3 = xy^3xy^{-3}$, is easily expressible as a product $w_3w_1^{-1}w_2^{-1}w_4$; this is left as an exercise for the reader. In particular, the above generators and single defining relation for Ψ' show that the abelianisation $\Psi'/(\Psi')'$ of Ψ' is free abelian of rank 4.

Now let us move to the quotient $\Psi/(\Psi')'$ of Ψ, which we will call G, and denote its derived subgroup $\Psi'/(\Psi')'$ by K. Then G is an extension of the free abelian subgroup $K \cong \mathbb{Z}^4$ by $G/K \cong C_2 \times C_6$. Also (for notational convenience) let us keep the same symbols x and y as the generators of G, and the same symbols w_1 to w_4 as the generators of K.

Then the action of the generators x and y by conjugation on the generators w_i of K may be given as follows:

$$\begin{aligned} {w_1}^x &= y^{-1}xyx = w_1^{-1}, \qquad & {w_2}^x &= yxy^{-1}x = w_2^{-1}, \\ {w_3}^x &= y^{-2}xy^2x = w_3^{-1}, \qquad & {w_4}^x &= y^2xy^{-2}x = w_4^{-1}, \end{aligned}$$

and

$$\begin{aligned} {w_1}^y &= y^{-1}xy^{-1}xy^2 = (y^{-1}xyx)(xy^{-2}xy^2) = w_1^{-1}w_3, \\ {w_2}^y &= y^{-1}xyx = w_1^{-1}, \\ {w_3}^y &= y^{-1}xy^{-2}xy^3 = (y^{-1}xyx)(xy^{-3}xy^3) = w_1^{-1}w_3w_1^{-1}w_2^{-1}w_4, \\ {w_4}^y &= y^{-1}xy^2xy^{-1} = (y^{-1}xyx)(xyxy^{-1}) = w_1^{-1}w_2. \end{aligned}$$

Accordingly, the generators x and y induce linear transformations of the free abelian group $K \cong \mathbb{Z}^4$ as follows:

$$x \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad y \mapsto \begin{pmatrix} -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ -2 & -1 & 1 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix}.$$

These matrices generate a group isomorphic to $G/K \cong C_2 \times C_6$,

They can be reduced mod m for any positive integer m, giving the corresponding action of x and y on the group $K/K^{(m)}$, where $K^{(m)}$ is the (characteristic) subgroup of K generated by the mth powers of all the w_i. Such actions can be used to consider subgroups of finite index in K that are invariant under the action of x and y (or in other words, subgroups of K that are normal in G with finite quotient), just as we did recently in a paper on arc-transitive abelian regular covers of cubic graphs [6].

In particular, when m is odd, we can change the basis of K from $\{w_1, w_2, w_3, w_4\}$ to $\{z_1, z_2, z_3, z_4\}$ where

$$z_1 = w_1w_4^{-1}, \quad z_2 = w_2w_3^{-1}, \quad z_3 = w_1w_2^{\frac{m+1}{2}}w_3^{\frac{m-1}{2}}, \quad z_4 = w_1^{\frac{m+1}{2}}w_2w_4^{\frac{m-1}{2}},$$

and get new matrices representing x and y, as follows:

$$x \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad y \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

This shows that the group $K/K^{(m)}$ can be expressed as the direct sum of two G-invariant subgroups of rank 2, say U and V, generated by the images of $\{z_1, z_2\}$ and $\{z_3, z_4\}$ respectively. Factoring out U or V gives a quotient of G of order $12m^2$, which is the rotation group of an orientable-regular map of type $\{6, 6\}$, characteristic $12m^2/6 - 12m^2/2 + 12m^2/6 = -2m^2$ and genus $m^2 + 1$.

Now the automorphism of G that inverts x and y clearly interchanges $w_1 = xy^{-1}xy$ with $xyxy^{-1} = w_2$, and similarly interchanges w_3 with w_4, and hence interchanges z_1 with z_2, and z_3 with z_4, and it follows that each of the two resulting maps is reflexible.

One of these maps has simple underlying graph, while the other does not. In the quotient obtained by factoring U, the element $z_1 = w_1w_4^{-1} = xy^{-1}xy(xy^2xy^{-2})^{-1} = xy^{-1}xy^3xy^{-2}x$ becomes trivial, and forces $1 = xy^3xy^{-2}xxy^{-1} = xy^3xy^{-3}$, so that y^3 is centralised by x, and hence the map obtained by factoring out U has multiple edges. On the other hand, this does not happen in the quotient obtained by factoring V. For example, when $m = 3$, these maps are R10.15 (for U) and its dual (for V).

In fact it is not difficult to see that U and V are interchanged by an automorphism that interchanges xy with y^{-1}, corresponding to map duality.

But the story does not end here. For some values of m, the G-invariant subgroups U and V are reducible, and we can factor out a larger (or smaller) subgroup L of $K/K^{(m)}$, and get the rotation group of an orientable-regular map of type $\{6, 6\}$ with simple underlying graph. Similarly for m even, we can find other kinds of non-trivial proper G-invariant subgroups of $K/K^{(m)}$, and factor out those. Examples obtainable in this way for which both the primal and dual maps have simple underlying graph include R10.13, C17.3, R17.20, C22.2, R28.9, R37.23, C40.2, C49.4, R49.36, R49.37, C50.3 and C50.4 (see [3]), with the rotation groups of these examples being isomorphic to extensions by $G/K \cong C_2 \times C_6$ of $(C_3)^2$, $(C_4)^2$, $(C_2)^4$, C_{21}, $(C_3)^3$, $(C_6)^2$, C_{39}, $C_4 \times C_{12}$, $C_4 \times C_{12}$, $(C_2)^3 \times C_6$, $C_7 \times C_7$ and $C_7 \times C_7$, respectively.

5 Main theorem

The families of orientable regular maps with simple underlying graphs presented in Section 3 give us the following:

Theorem 5.1 *For every positive integer $g \equiv 0, 1, 3, 4$ or 5 mod 6, there exists at least one orientably-regular map of genus g with simple underlying graph.*

Note that the family D (presented in sub-section 3.4) did not include a map of genus 4, but the map R4.2 of type $\{4, 5\}$ in [4] has simple underlying graph.

The above theorem covers 5/6 of all genera — indeed all except those congruent to 2 mod 6. Various families of maps of type $\{6, 6\}$ cover some of the remaining genera, as described in Section 4. It is also clear from the computational data obtained by the first author (see [3]) that there are numerous other families and examples.

In fact the first author has recently extended the determination of all orientably-regular maps up to genus 301, and there are no gaps at all in this range; in other words, for every positive integer $g \leq 301$, there exists at least one orientably-regular map of genus g with simple underlying graph.

For g in the range $2 \leq g \leq 301$ that are congruent to 2 mod 6, there is an orientably-regular map of type $\{6,6\}$ with simple underlying graph except when $g = 2$, 86, 116, 146, 188, 206, 236, 254, 266 or 296, and in all those cases, there are maps of other types with simple underlying graph — such as the duals of R2.1 (of type $\{8,3\}$) and R86.4 (of type $\{20,6\}$).

This gives us some confidence to make the following conjecture, although the question of how to prove it remains wide open:

Conjecture 5.2 *For every non-negative integer g, there exists at least one orientably-regular map of genus g with simple underlying graph.*

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