On the k-path vertex cover of some graph products

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Abstract

A subset S of vertices of a graph G is called a k-path vertex cover if every path of order k in G contains at least one vertex from S. Denote by $\psi_k(G)$ the minimum cardinality of a k-path vertex cover in G. In this paper improved lower and upper bounds for ψ_k of the Cartesian and the direct product of paths are derived. It is shown that for ψ_3 those bounds are tight. For the lexicographic product bounds are presented for ψ_k, moreover ψ_2 and ψ_3 are exactly determined for the lexicographic product of two arbitrary graphs. As a consequence the independence and the dissociation number of the lexicographic product are given.

Keywords: k-path vertex cover, vertex cover, dissociation number, independence number, graph products
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1. Introduction

For a graph G and a positive integer k, the subset $S \subseteq V(G)$ is a k-path vertex cover of G, if every path of order k in graph G contains a vertex from

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S. The cardinality of a minimum k-path vertex cover is denoted by $\psi_k(G)$. We say that a vertex is covered (uncovered) if it belongs (does not belong) to S.

The motivation for this invariant, which was introduced in [4], arises from communications in wireless sensor networks, where the data integrity is ensured by using the Novotný's k-generalized Cavas scheme [11]. Another motivation is in traffic control as presented in [14].

It is shown [4] that the problem of computing $\psi_k(G)$ is in general NP-hard for each $k \geq 2$, but polynomial for trees.

One way to look at the k-path vertex cover is as a generalization of the vertex cover. Note that $\psi_2(G)$ is equal to the size of a minimum vertex cover, moreover

$$\psi_2(G) = |V(G)| - \alpha(G),$$

where $\alpha(G)$ is the independence number of G. This gives an interesting connection to the well studied independence number [8, 9, 15, 13].

Also, the concept of the *dissociation number* of a graph [16] is in relation to the value of $\psi_3(G)$. A subset of vertices in a graph G is called a *dissociation set* if it induces a subgraph with maximum degree 1. The number of vertices in a maximum cardinality dissociation set in G is called the dissociation number of G and is denoted by $\mathrm{diss}(G)$. The relation between $\psi_3(G)$ and $\mathrm{diss}(G)$ is

$$\psi_3(G) = |V(G)| - \mathrm{diss}(G).$$

Determining the dissociation number of a graph is shown to be NP-hard in the class of bipartite graphs [16]. The dissociation number problem was also studied in several papers [1, 2, 5, 7], see [12] for a survey. A 2-approximation algorithm for 3-path vertex cover problem (for the weighted case of the problem) was presented by Tu and Zhou in [14]. In [10] an exact algorithm for computing $\psi_3(G)$ in running time $O(1.5171^n)$ for a graph of order n was presented.

Recently [3] it was shown that for an arbitrary graph G of order n and size m, with $1 \leq k \leq \frac{m}{n} \leq k+1$, the following holds $\psi_3(G) \leq \frac{kn}{k+2} + \frac{m}{(k+1)(k+2)}$. Some results on d-regular graphs are also presented, for instance for an arbitrary integer $k \geq 2$ and d-regular graph G, $d \geq k-1$, we have $\psi_k(G) \geq \frac{d-k+2}{2d-k+2}|V(G)|$.

2. Preliminaries and known results

Recall that the *Cartesian product* $G \,\square\, H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices $(u, v), (x, y)$ are adjacent whenever $u = x$ and $vy \in E(H)$, or $ux \in E(G)$ and $v = y$.

The *strong product* $G \boxtimes H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices $(u, v), (x, y)$ are adjacent whenever $u = x$ and $vy \in E(H)$, or $ux \in E(G)$ and $v = y$, or $ux \in E(G)$ and $vy \in E(H)$.

The *lexicographic product* $G \circ H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices $(u, v), (x, y)$ are adjacent whenever $ux \in E(G)$, or $u = x$ and $vy \in E(H)$.

Let G and H be arbitrary graphs, and $v \in V(H)$. We refer to the set $V(G) \times \{v\}$ as G-layer. Similarly $\{u\} \times V(H)$, $u \in V(G)$ is an H-layer. When referring to a specific G or H layer, we denote them by G^v or ${}^uH$, respectively. Layers can also be regarded as the graphs induced on these sets. Obviously, in the Cartesian, strong and lexicographic products, a G-layer or H-layer is isomorphic to G or H, respectively.

Since the next section deals with products of paths, we state the following formula for ψ_k of paths. For the path P_n on n vertices, the value of $\psi_k(P_n) = \lfloor \frac{n}{k} \rfloor$.

The following theorem, containing the formulas for ψ_3 of the Cartesian product of paths, was presented in [3].

Theorem 2.1. *(i)* $\psi_3(P_{2n+1} \,\square\, P_{2k}) = 2nk + \lfloor \frac{2k}{3} \rfloor$*, where* $n, k \geq 1$*,*
(ii) $\psi_3(P_{2n} \,\square\, P_{2k}) = 2nk$*, where* $n, k \geq 1$*,*
(iii) $\psi_3(P_{2n+1} \,\square\, P_{2k+1}) = n(2k+1) + \lfloor \frac{2k+1}{3} \rfloor$*, where* $1 \leq n \leq k$*.*

For an arbitrary k the following results are given.

Lemma 2.1. *[3] For each* $k \geq 4$, $\psi_k(P_{2\lceil \sqrt{k} \rceil} \,\square\, P_{3\lceil \sqrt{k} \rceil}) \geq \left\lceil \sqrt{k} \right\rceil$.

Proposition 2.1. *[3] For* $k \geq 4$, $n \geq 2\left\lceil \sqrt{k} \right\rceil$, $m \geq 3\left\lceil \sqrt{k} \right\rceil$*, the following holds*

$$\frac{nm}{24\left\lceil \sqrt{k} \right\rceil} \leq \psi_k(P_n \,\square\, P_m).$$

Proposition 2.2. *[3] For $k \geq 4$ the following holds*

$$\psi_k(P_n \,\Box\, P_m) \leq \frac{2nm}{\left\lfloor \sqrt{k} \right\rfloor} - \frac{2nm}{k}.$$

In this paper we present results on the ψ_k on several graph products. In the next section we improve the previously stated bounds for the Cartesian product of paths and extend these to the strong product of paths. In the last section the results on the lexicographic product of arbitrary graphs are presented. Among the upper and lower bounds for ψ_k, the exact values for ψ_2 and ψ_3 are determined. As a corollary of these results, a new proof for the independence number of the lexicographic product of arbitrary graphs is stated.

3. The Cartesian and the strong product

Before we present a new upper bound for $\psi_k(P_n \,\Box\, P_m)$, we will introduce some notions. Let D_i denote the set of all divisors of i. Choose $a, b \in D_i$, where $a \leq b$, in such way that $a \cdot b = i$ and the sum $a + b$ is the smallest possible. Note, that a is the largest element of D_i smaller or equal to $\sqrt{i}$, and b is the smallest element of D_i larger or equal to $\sqrt{i}$. We will call the pair (a, b) the *middle D_i pair*. The importance of $a + b$ being the smallest possible is evident, since the number of covered vertices depends on this sum, so taking other pairs $(a', b') \in D_i$, $a' \cdot b' = i$, of divisors would give a worse bound.

Proposition 3.1. *Let $k \geq 3$ and (a, b) be the middle D_{k-1} pair. Then the following holds*

$$\begin{aligned} \psi_k(P_n \,\Box\, P_m) \;\; \leq \min \;\; & \left\{ \left\lfloor \frac{n}{a+1} \right\rfloor m + \left\lfloor \frac{m}{b+1} \right\rfloor n - 2 \left\lfloor \frac{n}{a+1} \right\rfloor \left\lfloor \frac{m}{b+1} \right\rfloor , \right. \\ & \left. \left\lfloor \frac{n}{b+1} \right\rfloor m + \left\lfloor \frac{m}{a+1} \right\rfloor n - 2 \left\lfloor \frac{n}{b+1} \right\rfloor \left\lfloor \frac{m}{a+1} \right\rfloor \right\} \end{aligned}$$

Proof. We will construct a k-path vertex cover with at most $\lfloor \frac{n}{a+1} \rfloor m + \lfloor \frac{m}{b+1} \rfloor n - 2 \lfloor \frac{n}{a+1} \rfloor \lfloor \frac{m}{b+1} \rfloor$ vertices.

Let $S_1 = \{(i, j) \in P_n \,\Box\, P_m \mid i \equiv 0 \pmod{a+1}\}$ (for all applicable indicies i and j) and similarly $S_2 = \{(i, j) \in P_n \,\Box\, P_m \mid j \equiv 0 \pmod{b+1}\}$. It is easy to see that $S = (S_1 \cup S_2) \backslash (S_1 \cap S_2)$ is a k-path vertex cover, since the largest connected subgraph of $P_n \,\square\, P_m$ with all vertices uncovered is isomorphic to $P_a \,\square\, P_b$. The constructed k-path vertex cover can be seen in Fig. 1.

In a P_n-layer we cover each $(a+1)$-st vertex, since there are m such layers, the size of S_1 is at most $|S_1| \leq \frac{nm}{a+1}$. Similarly, $|S_2| \leq \frac{nm}{b+1}$. The vertices $(i,j) \in S_1 \cap S_2$ can be left uncovered, because all the vertices $(i \pm 1, j)$ and $(i, j \pm 1)$ are in S. Since the size of $|S_1 \cap S_2| \leq \lfloor \frac{n}{a+1} \rfloor \lfloor \frac{m}{b+1} \rfloor$ and we counted every vertex in the intersection twice, the size of S is $|S| \leq \lfloor \frac{n}{a+1} \rfloor m + \lfloor \frac{m}{b+1} \rfloor n - 2 \lfloor \frac{n}{a+1} \rfloor \lfloor \frac{m}{b+1} \rfloor$.

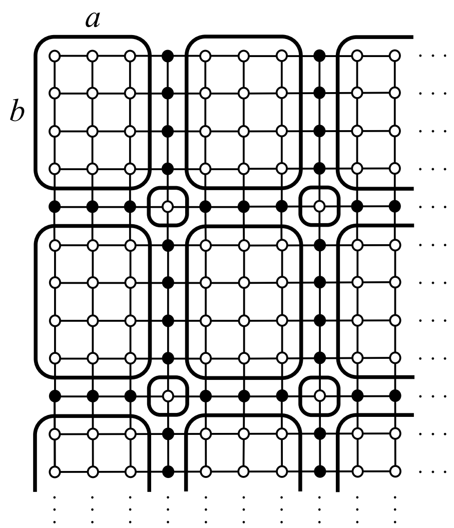

Figure 1: A k-path vertex cover of $P_n \,\square\, P_m$.

Similarly, one can construct a k-path vertex cover with at most $\lfloor \frac{n}{b+1} \rfloor m + \lfloor \frac{m}{a+1} \rfloor n - 2 \lfloor \frac{n}{b+1} \rfloor \lfloor \frac{m}{a+1} \rfloor$ vertices. The assertion then follows immediately. $\square$

Note, that for $k = 3$ this bound is sharp, since using the middle D_2 pair and the above procedure, the described k-path vertex cover corresponds to the one from Theorem 2.1, presented in [3]. Note that the bound from Proposition 2.2 for $k = 3$ is worse than the presented improved result, since it states we need to cover all the vertices.

The same approach as in Proposition 3.1 gives us an upper bound for the strong product of graphs.

Proposition 3.2. *Let $k \geq 3$ and (a, b) be the middle D_{k-1} pair. Then the following holds*

$$\psi_k(P_n \boxtimes P_m) \leq \min\left\{ \left\lfloor \frac{n}{a+1} \right\rfloor m + \left\lfloor \frac{m}{b+1} \right\rfloor n, \left\lfloor \frac{n}{b+1} \right\rfloor m + \left\lfloor \frac{m}{a+1} \right\rfloor n \right\}.$$

Proof. We will construct a k-path vertex cover with at most $\lfloor \frac{n}{a+1} \rfloor m + \lfloor \frac{m}{b+1} \rfloor n$ vertices in the same manner as in the proof of Proposition 3.1.

Let $S_1 = \{(i, j) \in P_n \boxtimes P_m \mid i \equiv 0 \pmod{a+1}\}$ (for all applicable indicies i and j) and similarly $S_2 = \{(i, j) \in P_n \boxtimes P_m \mid j \equiv 0 \pmod{b+1}\}$. It is easy to see that $S = S_1 \cup S_2$ is a k-path vertex cover (see Fig. 2), since the largest connected subgraph of $P_n \boxtimes P_m$ with all vertices uncovered is isomorphic to $P_a \boxtimes P_b$. Note that we cannot leave the vertices in $S_1 \cap S_2$ uncovered (as in the case of $P_n \square P_m$) due to *diagonal* edges in the strong product.

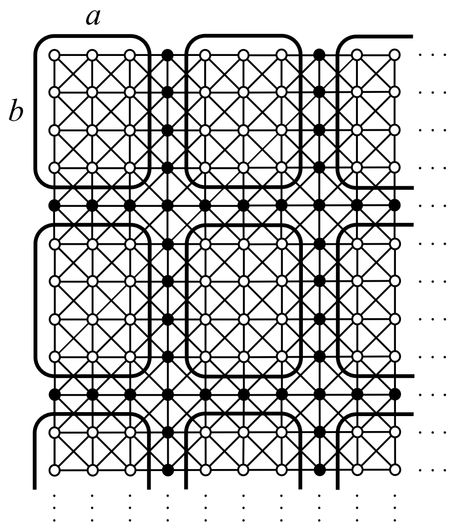

Figure 2: A k-path vertex cover of $P_n \boxtimes P_m$.

Similarly, one can construct a k-path vertex cover with at most $\lfloor \frac{n}{b+1} \rfloor m + \lfloor \frac{m}{a+1} \rfloor n$ vertices. Following the same line of thought as in the proof of Proposition 3.1 the assertion follows. $\square$

As for a lower bound for the strong product of paths, we will first prove the following Lemma.

Lemma 3.1. *Let $k \geq 4$ and let (a, b) be the middle D_{k-1} pair. Then $\psi_k(P_{2b} \boxtimes P_{b+1}) \geq b + 1$.*

Proof. Assume to the contrary that S is a k-path vertex cover of the graph $P_{2b} \boxtimes P_{b+1}$, with $|S| \leq b$. Then $P_{2b} \boxtimes P_{b+1}$ has at least b of all P_{b+1}-layers not containing any vertex of S. All of the P_{b+1}-layers that do have at least one vertex in S, also have at least one vertex not in S. Then there exists such a vertex $v \notin S$ in the layer $P_{b+1}^{u_i}$, where $1 < i < 2b$, that one can connect this vertex with a vertex $v' \notin S$ in the neighboring layers $P_{b+1}^{u_{i\pm 1}}$. Now, using the P_{b+1}-layers not containing any vertex of S and moving from/to the other layers on uncovered vertices only, one can easily construct a path on at least $b \cdot b + 1$ vertices, not containing any vertex of S. Since $b \cdot b + 1 \geq ab + 1 = ab + 1 = k - 1 + 1 = k$, we have a path of order at least k without any vertices in S, which is a contradiction to the assumption that S is a k-path vertex cover. □

Proposition 3.3. *Let $k \geq 4$, let (a, b) be the middle D_{k-1} pair and $n \geq 2b, m \geq b + 1$. Then the following holds*

$$\frac{nm}{8b} \leq \psi_k(P_n \boxtimes P_m).$$

Proof. We split the whole graph $P_n \boxtimes P_m$ into r disjoint subgraphs isomorphic to $P_{2b} \boxtimes P_{b+1}$ such that $r(2b)(b + 1) \geq \frac{1}{4}nm$. By Lemma 3.1 a k-path vertex cover must have at least $b+1$ vertices in each subgraph isomorphic to $P_{2b} \boxtimes P_{b+1}$ in G, hence:

$$\psi_k(P_n \boxtimes P_m) \geq r(b + 1) \geq \frac{nm}{8b}.$$

□

4. The lexicographic product

As seen in the previous section it is hard to determine exact results even for fixed graphs G and H. In this section we give more general results for the lexicographic product of graphs. It turns out that many of this results are a generalization of some previously known results for other invariants.

Proposition 4.1. *Let G and H be two arbitrary graphs. Then*

$$\psi_k(G \circ H) \leq |V(G)||V(H)| - (|V(G)| - \psi_2(G))(|V(H)| - \psi_k(H)).$$

Proof. Let $I = \{v_1, \ldots, v_{\alpha(G)}\}$ be a maximum independent set of graph G, $J = V(G)\backslash I = \{v_{\alpha(G)+1}, \ldots, v_{|V(G)|}\}$. Denote by S_i the set of covered vertices in the subgraph $H_i = \{v_i\} \circ H$. Cover the subgraph H_i, $i \in \{1, \ldots, \alpha(G)\}$, with exactly $\psi_k(H)$ vertices and cover the subgraph H_j, $j \in \{\alpha(G)+1, \ldots, |V(G)|\}$, with exactly $|V(H)|$ vertices, hence $|S_i| = \psi_k(H)$, $i \in \{1, \ldots, \alpha(G)\}$, and $|S_j| = |V(H)|$, $j \in \{\alpha(G)+1, \ldots, |V(G)|\}$. This cover is by definition a proper k-path vertex cover of graph $G \circ H$. It follows that

$$\begin{aligned}
\psi_k(G \circ H) &\leq \sum_{i=1}^{|V(G)|} |S_i| = \alpha(G)\psi_k(H) + (|V(G)| - \alpha(G))|V(H)| \\
&= |V(G)||V(H)| + \alpha(G)\psi_k(H) - \alpha(G)|V(H)| \\
&= |V(G)||V(H)| - \alpha(G)(|V(H)| - \psi_k(H)) \\
&= |V(G)||V(H)| - (|V(G)| - \psi_2(G))(|V(H)| - \psi_k(H))
\end{aligned}$$

□

Theorem 4.1. *Let G be an arbitrary graph and H a graph different from the vertex graph. Then*

$$\psi_3(G \circ H) = |V(G)||V(H)| - (|V(G)| - \psi_2(G))(|V(H)| - \psi_3(H))\,.$$

Proof. Let S be the optimal 3-path vertex cover of graph $G \circ H$ and $S_i \subseteq S$ the set of covered vertices in the subgraph $\{v_i\} \circ H$, $i \in \{1, \ldots, |V(G)|\}$. Hence $|S| = \sum_{i=1}^{|V(G)|} |S_i|$. Let $H_1, H_2, \ldots, H_l$ be those H-layers that contain uncovered vertices and T_i the set of uncovered vertices in H_i, $i \in \{1, \ldots, l\}$. It is obvious that $|T_i| \geq 1$ for all i. We consider two cases.

Case 1: let $|T_i| \geq 2$ for all i. Then the subgraph $H_i = \{v_i\} \circ H$ has at least two vertices that are not in S. The neighboring H-layers of layer H_i must have all its vertices in S as seen in Fig. 3.

Note that the maximum number of all such T_i sets equals to the independence number of G, $\alpha(G)$. Therefore S must contain at least $\alpha(G)\psi_3(H) + (|V(G)| - \alpha(G))|V(H)|$ vertices in this optimal 3-path vertex cover, hence

$$\begin{aligned}
\psi_3(G \circ H) &\geq \alpha(G)\psi_3(H) + (|V(G)| - \alpha(G))|V(H)| \\
&= |V(G)||V(H)| - (|V(G)| - \psi_2(G))(|V(H)| - \psi_3(H))
\end{aligned}$$

According to Proposition 4.1 this is also the upper bound and therefore

$$\psi_3(G \circ H) = |V(G)||V(H)| - (|V(G)| - \psi_2(G))(|V(H)| - \psi_3(H))\,.$$

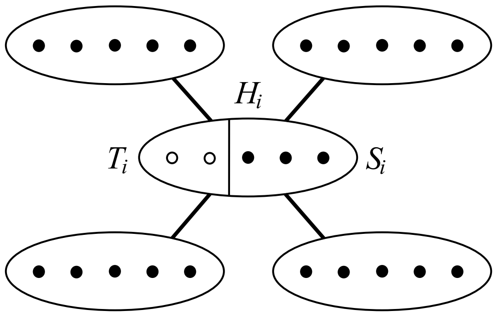

Figure 3: Two uncovered vertices in H_i

Case 2: Let $|T_i| = 1$ for some i. Then one of the neighboring H-layers of H_i, say layer H_j, must also contain exactly one uncovered vertex, otherwise S would not be an optimal 3-path vertex cover since we could without extra cost uncover another vertex in layer H_i and make a better 3-path vertex cover. Moreover every other neighboring layer of both layers H_i and H_j must contribute all their verties to S otherwise S would not be a proper 3-path vertex cover, see Fig. 4. Now we cover the only uncovered vertex in layer H_j and uncover an arbitrary covered vertex in layer H_i. This is possible since graph H (therefore also H_i) has at least two vertices. In this way we get another optimal 3-path vertex cover of $G \circ H$ and moreover H_i now has two uncovered vertices, hence $|T_i| = 2$. Next we move to another H_i layer that has only one uncovered vertex and repeat the above procedure. In this way we end up with Case 1 which proves the theorem. $\square$

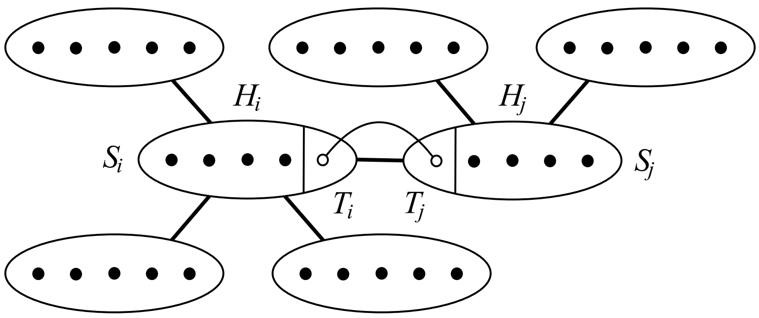

Figure 4: One uncovered vertex in H_i

For $H = K_1$ the solution is trivial. According to the theorem above we would get $\psi_3(G \circ K_1) = |V(G)| - (|V(G)| - \psi_2(G)) = \psi_2(G)$ which in general is not true, since $\psi_3(G \circ K_1) = \psi_3(G) \leq \psi_2(G)$.

One would think of a similar theorem for general k: Let G be an arbitrary graph and H a graph with $|V(H)| \geq k-1$. Then

$$\psi_k(G \circ H) = |V(G)||V(H)| - (|V(G)| - \psi_2(G))(|V(H)| - \psi_k(H)) \,.$$

But the above assertion is not true already for $k = 4$. Take for example $G = P_3$ and for H the independent set on n vertices, S_n. Then $\psi_4(P_3 \circ S_n) = n-1$ which is less then $|V(P_3)||V(S_n)| - (|V(P_3)| - \psi_2(P_3))(|V(S_n)| - \psi_4(S_n)) = 3n - 2n = n$

We get a nice corollary from Theorem 4.1 which gives the exact value for the dissociation number of the lexicographic product of two arbitrary graphs.

Corollary 4.1. *Let G and H be two arbitrary graphs. Then*

$$\mathrm{diss}(G \circ H) = \left\{ \begin{array}{rl} \alpha(G)\mathrm{diss}(H) & \textit{for} \quad H \neq K_1 \\ \mathrm{diss}(G) & \textit{for} \quad H = K_1 \end{array} \right. .$$

Proof. Let $H \neq K_1$. For any two graphs G and H it follows that $\mathrm{diss(G \circ H)} = \mathrm{|V(G)||V(H)|} - \psi_3(\mathrm{G \circ H})$. The assertion follows immediately from Theorem 4.1. If $H = K_1$ the result is trivial. □

Next we introduce some lower bounds for ψ_k of the lexicographic product of two arbitrary graphs. The following Proposition is straight forward to prove.

Proposition 4.2. *Let G and H be two arbitrary graphs. Then*

$$\psi_k(G \circ H) \geq |V(G)|\psi_k(H) \,.$$

Proof. In every H-layer we need at least $\psi_k(G)$ vertices covered. There are exactly $|V(G)|$ such layers. Therefore we need at least $|V(G)|\psi_k(H)$ covered vertices in every k-path vertex cover of graph $G \circ H$. □

The trivial lower bound proven in Proposition 4.2 is tight for the graph on n independent vertices S_n. It is easy to show that $\psi_k(S_n \circ H) = |V(G)|\psi_k(H)$ for any $k \geq 2$. One can also see that the equality is achived for some graphs G with $\psi_k(G) = 0$.

We can indeed find a lower bound which improves the trivial bound.

Theorem 4.2. *Let G and H be two arbitrary graphs. Then*

$$\psi_k(G \circ H) \geq |V(G)||V(H)| - (|V(G)| - \psi_k(G))(|V(H)| - \psi_k(H)) .$$

Proof. Let S be a k-path vertex cover of graph $G \circ H$ and $S_i \subseteq S$ the set of covered vertices in the subgraph $\{v_i\} \circ H$, $i \in \{1, \ldots, |V(G)|\}$, hence $|S| = \sum_{n=1}^{|V(G)|} |S_i|$.

It is clear that $\psi_k(H_i) \leq |S_i| \leq |V(H_i)|$ for any $i \in \{1, \ldots, |V(G)|\}$. We consider two cases.

Case 1: If $|S_i| < |V(H_i)|$, for all i, then each H-layer has at least one uncovered vertex in S. This means that we can find a subgraph G' of graph $G \circ H$, which consists only of uncovered vertices, each vertex being in their own H-layer of graph $G \circ H$, and is isomorphic to G, see Fig. 5. Since S

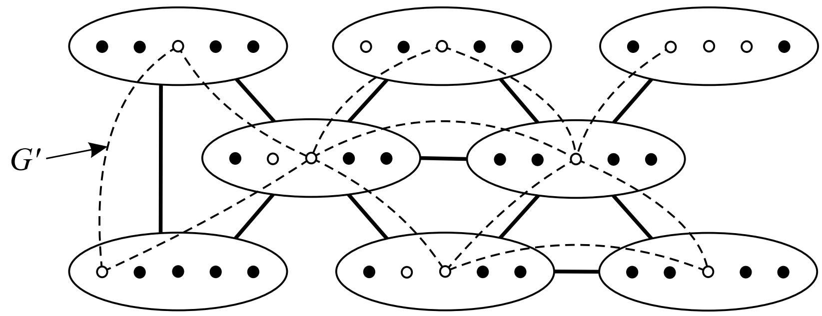

Figure 5: Subgraph G' with uncovered vertices

is a k-path vertex cover $\psi_k(G)$ must be zero. Then the lower bound holds immediately according to Proposition 4.2.

Case 2: Let $|S_i| = |V(H_i)|$ for some i and let $H_1, \ldots, H_l$ be those H-layers for which this equality holds. We can find a subgraph G'' of graph $G \circ H$, which is isomorphic to graph G, has each of its vertices in their own H-layer of graph $G \circ H$, and has exactly l covered vertices, see Fig. 6. This means that $l \geq \psi_k(G)$ otherwise S would not be a k-path vertex cover. Then we have exactly l such H-layers that have all their vertices in S and

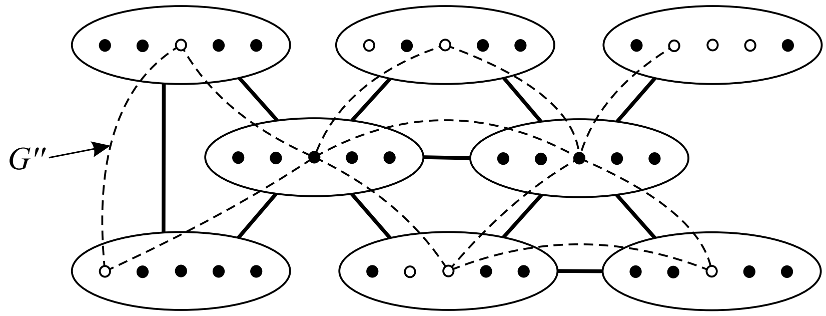

Figure 6: Subgraph G'' with some covered vertices

$(|V(G)| - l)$ such H-layers that have at least $\psi_k(H)$ vertices in S. Hence

$$\begin{aligned}
|S| &\geq l|V(H)| + (|V(G)| - l)\psi_k(H) \\
&= |V(G)|\psi_k(H) + l(|V(H)| - \psi_k(H)) \\
&\geq |V(G)|\psi_k(H) + \psi_k(G)(|V(H)| - \psi_k(H)) \\
&= |V(G)|\psi_k(H) + \psi_k(G)|V(H)| - \psi_k(G)\psi_k(H) \\
&= |V(G)||V(H)| - (|V(G)| - \psi_k(G))(|V(H)| - \psi_k(H)).
\end{aligned}$$

□

Using Proposition 4.1 and Theorem 4.2 we get the following proposition.

Proposition 4.3. *Let G and H be two arbitrary graphs. Then*

$$\psi_2(G \circ H) = |V(G)||V(H)| - (|V(G)| - \psi_2(G))(|V(H)| - \psi_2(H)) \, .$$

Proposition 4.3 immidiately implies a well known result of Geller and Stahl (see [6]) who determined the independence number of the lexicographic product.

Corollary 4.2. *Let G and H be two arbitrary graphs. Then*

$$\alpha(G \circ H) = \alpha(G)\alpha(H) \, .$$

Proof. For any graphs G and H it follows that $\alpha(G \circ H) = |V(G)||V(H)| - \psi_2(G \circ H)$. The assertion follows immediately from Proposition 4.3. □

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