Sierpiński graphs as spanning subgraphs of Hanoi graphs

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December 21, 2011

Abstract

Hanoi graphs H_p^n model the Tower of Hanoi game with p pegs and n discs. Sierpiński graphs S_p^n arose in investigations of universal topological spaces and have meanwhile been studied extensively. It is proved that S_p^n embeds as a spanning subgraph into H_p^n if and only if p is odd or, trivially, if $n = 1$.

Keywords: Sierpiński graph; Hanoi graph; spanning subgraph; Hamming graph

AMS Subj. Class. (2010): 05C60

1 Introduction

The Hanoi graphs H_p^n form a natural mathematical model for the Tower of Hanoi game with p pegs and n discs. The puzzle with three pegs is well understood, cf. [5]. Surprisingly, even the simplest task—to move a perfect tower of discs to another perfect tower in an optimal number of moves—presents a notorious open problem for four or more pegs, cf. [1, 3, 11]. This means that the distance function on the graphs H_p^n, $p \geq 4$, is far from being well understood. Some other properties of Hanoi graphs are less difficult to access. For instance, Hanoi graphs were classified with respect to planarity [6, Theorem 2]; they are in edge- and total coloring class 1, except those isomorphic to a complete graph of odd or even order, respectively [7, Theorems 3 and 4]; the automorphism group of H_p^n is isomorphic to the symmetric group on $[p]_0$, induced by the permutations of pegs [15, Main Theorem].

Sierpiński graphs S_p^n, introduced and studied for the first time in [10], were in part motivated by investigations of certain universal topological spaces [14]. (See the recent book of Lipscomb [13] for the state of the art about these spaces.) It was shown in [10, Theorem 2] that S_3^n is isomorphic to H_3^n for any n. In other words, both graphs can be represented by the same drawing but with different labellings. This difference allows two approaches to the Tower of Hanoi; a beautiful example for using the Sierpiński labelling is due to Romik [17]. Sierpiński graphs have been extensively studied by now; see, e.g., recent papers [2, 4, 7, 8, 12, 16] and references therein. For instance, the hub number of Sierpiński graphs was determine in [12, Theorem 9] and their average eccentricity in [8, Corollary 3.5].

Although for any $p, n \in \mathbb{N}$, the graphs S_p^n and H_p^n are defined on the same vertex set, they cannot be isomorphic anymore for $p > 3$ and $n > 1$; for instance, it will turn out that then $\|S_p^n\| < \|H_p^n\|$, where $\|G\|$ denotes the size of a graph G. Therefore it is natural to ask whether S_p^n can be embedded into H_p^n as a spanning subgraph. In this note we will answer that question exhaustively by proving that this holds if and only if p is odd. We hope that the result will lead to further insights into the mathematics of the Tower of Hanoi.

In the next section Sierpiński and Hanoi graphs will be defined, some of their properties recalled and some notation introduced. In Section 3 the main result of this note is proved and discussed.

2 Sierpiński and Hanoi graphs

Let $p, n \in \mathbb{N}$, then the *Sierpiński graph*, S_p^n is defined as follows. The vertex set of S_p^n is the set $[p]_0^n$, $[p]_0 := \{0, \ldots, p-1\}$, whose elements we will denote by $s_n \ldots s_1$ for simplicity. Two vertices s and t are adjacent if and only if there exists a $\delta \in [n] := \{1, \ldots, n\}$ such that

(i) $s_d = t_d$, for $d \in [n] \setminus [\delta]$;

(ii) $s_\delta \neq t_\delta$;

(iii) $s_d = t_\delta$ and $t_d = s_\delta$ for $d \in [\delta - 1]$.

For any n, $S_1^n \cong K_1$ and $S_2^n \cong P_{2^n}$. Note also that $S_p^1 \cong K_p$ for any p. For a drawing of the graph S_4^2 see Figure 1. Vertices of the form $k \ldots k = k^n$ are called *extreme vertices* of S_p^n. Clearly, S_p^n contains p extreme vertices and they are of degree $p-1$; all the other vertices are of degree p.

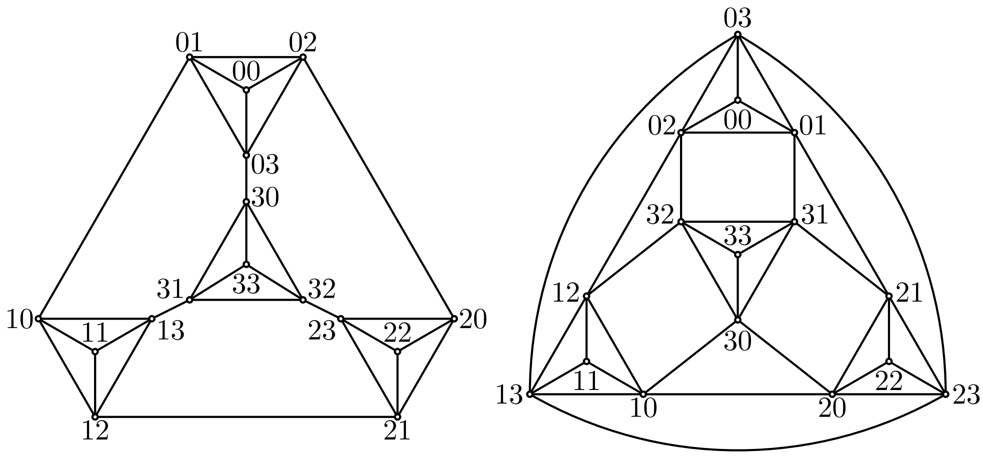

Figure 1: Sierpiński graph S_4^2 and Hanoi graph H_4^2

The *Tower of Hanoi* consists of p vertical pegs and n discs of mutually different diameters, each of which can be stacked onto one of the pegs. A distribution of all discs on the pegs with no larger disc lying on a smaller one is called a *regular state*. A *perfect state* is a regular state with all discs arranged on one and the same peg. A *legal move* is to move a disc from the top of a stack on one peg to the top of the (possibly empty) stack on another peg, provided both states involved are regular. Labelling pegs with numbers from $[p]_0$ and discs with numbers from $[n]$ in increasing order according to size, a regular state can be represented uniquely by a vector $s \in [p]_0^n$, which we will again write as $s_n \dots s_1$, and whose component s_d is the peg on which disc d is lying. The *Hanoi graph* H_p^n is then defined on the vertex set $[p]_0^n$, and two vertices (= two regular states) are adjacent if one can be obtained from the other by a legal move. Note that adjacent vertices of H_p^n differ in precisely one coordinate.

Again, $H_1^n \cong K_1$ for every n and $H_p^1 \cong K_p$ for any p. Every vertex (= regular state) of H_2^n is adjacent to exactly one vertex since only the smallest disc can move. Therefore, H_2^n is the disjoint union of 2^{n-1} copies of K_2. H_3^n is the state graph of the classical Tower of Hanoi. For a representation of the graph H_4^2 see Figure 1. Vertices of the form $k \dots k = k^n$ will be called *perfect vertices* of H_p^n. Note that H_p^n contains p perfect vertices and that they are all of degree $p-1$ because in a perfect state the only legal moves are moves of the smallest disc. Any other vertex of H_p^n has degree at least $2p-3$, because the second smallest disc in a top position on some peg can move to $p-2$ target pegs. (Note that there are no non-perfect vertices if $p=1$ or $n=1$.) This shows that $S_p^n \cong H_p^n$ if and only if $p \in \{1,3\}$ or $n=1$, because the maximal degree of

S_p^n is $p < 2p-3$ for $p > 3$ and $n > 1$.

Moreover, S_p^n and H_p^n can be viewed as constructed recursively with $S_p^1 = H_p^1$ and S_p^{1+n} and H_p^{1+n} composed from p copies iS_p^n and iH_p^n, respectively. The copies iS_p^n and jS_p^n are joined by the single edge $\{ij^n, ji^n\}$, whereas in the Tower of Hanoi all states with discs 1 to n not on pegs i and j allow for a move of the largest disc from i to j or vice versa. This leads to the recurrences

$$\|S_p^0\| = 0 = \|H_p^0\|\,,\ \|S_p^{1+n}\| = p\|S_p^n\| + \binom{p}{2}\,,\ \|H_p^{1+n}\| = p\|H_p^n\| + \binom{p}{2} \cdot (p-2)^n\,,$$

whence $\|S_p^n\| < \|H_p^n\|$ for $p > 3$ and $n > 1$.

We will consider the following subgraphs of S_p^n and H_p^n. Let $s_d \in [p]_0$ for $d \in [n]\setminus[r]$, $r \in [n-1]$; then $s_n \ldots s_{r+1}S_p^r$ and $s_n \ldots s_{r+1}H_p^r$ denote the subgraphs of S_p^n and H_p^n induced by vertices whose components s_{r+1} to s_n are fixed. Clearly, $s_n \ldots s_{r+1}S_p^r$ and $s_n \ldots s_{r+1}H_p^r$ are isomorphic to S_p^r and H_p^r, respectively.

A *clique* of a graph G is a complete subgraph of G maximal with respect to inclusion, i.e. not contained in any larger complete subgraph. A *q-clique* is a clique of order q. The *clique number* $\omega(G)$ is the order of a largest clique of G. By induction on n one can show that in S_p^n, $p \geq 3$, the only cliques are 2- and p-cliques. The p-cliques are just the subgraphs $s_n \ldots s_2S_p^1$; any edge not in these cliques induces a 2-clique. For the cliques of H_p^n we have:

Lemma 1 *Every complete subgraph of H_p^n, $p, n \in \mathbb{N}$, is induced by edges corresponding to moves of one and the same disc. In particular, $\omega(H_p^n) = p$ and $s_n \ldots s_2H_p^1$ are the only p-cliques of H_p^n.*

Proof. The cases $p = 1$ and $p = 2$ are trivial. For $p \geq 3$ take any vertex s joined to two vertices s' and s'' by edges corresponding to the moves of two different discs. Then the positions of these discs differ in s' and s''. Since vertices in H_p^n can only be adjacent if they differ in precisely one coordinate, s' and s'' cannot be adjacent. This proves the first assertion. In any state s, the smallest disc can move to $p-1$ pegs, so that s is contained in a p-clique. On the other hand, a disc $d \neq 1$ can be transferred to at most $p-2$ pegs, namely those not occupied by disc 1. $\square$

3 The main result

Theorem 2 *Let $p, n \in \mathbb{N}$. Then S_p^n can be embedded isomorphically into H_p^n if and only if p is odd or $n = 1$.*

Proof. The case $n = 1$ is clear, because $S_p^1 = H_p^1$. The same applies to $p = 1$ since $S_1^n = H_1^n$. Moreover, for $n \geq 2$, we have $\|S_2^n\| = 2^n - 1 > 2^{n-1} = \|H_2^n\|$, so that S_2^n

can not be embedded isomorphically into H_2^n. (In fact, in this case H_2^n is a spanning subgraph of S_2^n.)

Now let $p \geq 4$ be even and $n = 2$. Assume that $\alpha : S_p^2 \to H_p^2$ is an isomorphic embedding. By Lemma 1, the p-cliques of S_p^2 are mapped onto the p-cliques of H_p^2. The remaining edges of S_p^2, which are of the form $\{ij, ji\}$, $i \neq j$, have to be mapped by α to edges in H_p^2 corresponding to moves of disc 2. Note that these $\binom{p}{2}$ edges of S_p^2 are pairwise non-incident. On the other hand, edges in H_p^2 corresponding to moves of disc 2 induce p cliques of order $p-1$. Among the edges of these cliques, we can select at most $p\left\lfloor\frac{p-1}{2}\right\rfloor$ independent ones. Since p is even, $p\left\lfloor\frac{p-1}{2}\right\rfloor < p\,\frac{p-1}{2} = \binom{p}{2}$. We conclude that S_p^2 cannot be embedded isomorphically into H_p^2.

We will now reduce the more general case for even p, but with $n \geq 3$, to the case just dealt with by considering the image of subgraph $0^{n-2}S_p^2$ under an assumed isomorphic embedding α of S_p^n into H_p^n. By the degree condition, α maps extreme vertices onto perfect vertices, in particular, $\alpha(0^n) = j^n$ for some j. Using Lemma 1 again, $\alpha(0^{n-1}S_p^1) = j^{n-1}H_p^1$. Moreover, the subgraph $0^{n-2}S_p^2$ of S_p^n contains $p-1$ p-cliques that are at distance 1 from the clique $0^{n-1}S_p^1$. All the other cliques of S_p^n are at distance more than 1 from $0^{n-1}S_p^1$. Similarly, the subgraph $j^{n-2}H_p^2$ of H_p^n contains p p-cliques that are pairwise at distance 1. Every other clique of H_p^2 is at distance at least two from $j^{n-1}H_p^1$. Therefore, $\alpha(0^{n-2}S_p^2) = j^{n-2}H_p^2$. Hence α embeds $0^{n-2}S_p^2 \cong S_p^2$ isomorphically onto $j^{n-2}H_p^2 \cong H_p^2$, a fact which we already excluded.

Suppose next that $p \geq 3$ is odd. We will show by induction on n that there is an isomorphic embedding of S_p^n into H_p^n, the case $n = 1$ being trivial. By the degree condition, any such embedding must map extreme vertices of S_p^n onto perfect vertices of H_p^n. For $n \geq 1$ let ι_n be an isomorphic embedding from S_p^n onto H_p^n. Since an arbitrary permutation of the perfect states of H_p^n extends to an automorphism of H_p^n (cf. [15]), we may without loss of generality assume that $\iota_n(k^n) = k^n$ for all k. We construct the mapping $\iota_{1+n} : S_p^{1+n} \to H_p^{1+n}$ in the following way. For $k \in [p]_0$ define the permutation π_k on $[p]_0$ as follows:

$$\forall\, i \in [p]_0 : \ \pi_k(i) = \frac{1}{2}\left(k(p+1) - i(p-1)\right) \bmod p\,;$$

it has precisely one fixed point, namely k. Then let π_k^n denote the bijection on $[p]_0^n$ with $\pi_k^n(s_n \ldots s_1) = \pi_k(s_n) \ldots \pi_k(s_1)$. Define

$$\forall\, k \in [p]_0\ \forall\, s \in [p]_0^n : \ \iota_{1+n}(ks) = k\pi_k^n\left(\iota_n(s)\right)\,.$$

This obviously constitutes a bijection with

$$\iota_{1+n}(k^{1+n}) = k\pi_k^n\left(\iota_n(k^n)\right) = k\pi_k^n(k^n) = k^{1+n}\,.$$

It remains to show that $\{\iota_{1+n}(ij^n), \iota_{1+n}(ji^n)\} \in E(H_p^{1+n})$ for $i, j \in [p]_0$, $i \neq j$. We have $\iota_{1+n}(ij^n) = i\pi_i^n\left(\iota_n(j^n)\right) = i\pi_i(j)^n$ and similarly $\iota_{1+n}(ji^n) = j\pi_j(i)^n$. Moreover,

$$i \neq \pi_i(j) = \frac{1}{2}\left(ip + i - jp + j\right) \bmod p = \frac{1}{2}\left(jp + j - ip + i\right) \bmod p = \pi_j(i) \neq j\,,$$

and so the two vertices are adjacent in H_p^n. $\square$

Let $r_\ell \geq 2$, $\ell \in [n]$, be given integers. Let G be the graph whose vertices are $[r_1] \times [r_2] \times \cdots \times [r_n]$, two vertices being adjacent if the corresponding tuples differ in precisely one coordinate. Then G is called a *Hamming graph*. Alternatively, a Hamming graph is the Cartesian product graph $K_{r_1} \,\square\, K_{r_2} \,\square \cdots \square\, K_{r_n}$. As observed in [9, Section 2.2], Hanoi graphs H_p^n are spanning subgraphs of $K_p \,\square \cdots \square\, K_p = K_p^n$. Therefore, we get

Corollary 3 *Let p be odd. Then for any n, S_p^n is a spanning subgraph of the Hamming graph K_p^n.*

Acknowledgments

This work has been financed by ARRS Slovenia under the grant P1-0297 and within the EUROCORES Programme EUROGIGA/GReGAS of the European Science Foundation. The second author is also with the Institute of Mathematics, Physics and Mechanics, Ljubljana.

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